Original Research Article

Picard and Adomian solutions of a nonlocal Cauchy problem of a delay differential equation

Abstract

In this paper, two methods are used to solve a nonlocal Cauchy problem of a delay differential equation; Adomian decomposition method (ADM) and Picard method. The existence and uniqueness of the solution are proved. The convergence of the series solution and the error analysis are studied.

Keywords: Nonlocal Cauchy problem, existence, uniqueness, error analysis, Adomian method, Picard method.

MSC: [2000]34A12, 34A30, 34D20

1 Introduction

The Cauchy problems with multi-point or non-local conditions have been extensively studied by several authors in the last two decades [[1] - [14]]. Here we are concerned with the nonlocal Cauchy problem of the delay differential equation

$$\frac{dx(t)}{dt} = f(t, x(t-r)), \quad t \in (0, T] \tag{1}$$

$$x(t) = x_0, \quad t < 0, \tag{2}$$

with the nonlocal condition

$$x(0) = \sum_{k=1}^{n} a_k x(t_k), \quad t_k \in (r, T). \tag{3}$$

The existence and uniqueness of the solution $x \in C(J)$,where $C(J)$ is the space of all continuous functions and $J = [0, T]$, $T < \infty$ of the nonlocal problem (1)-(3) will be proved, the integral representation of this solution will be proved, the solution algorithm using ADM will be given and the converge of the series solution is proved.

2 Problem Solving

2.1 Integral representation

For the integral representation of the solution of the nonlocal problem (1)-(3) we have the following lemma.

Lemma 1 If $\left(1-\sum\limits_{k=1}^{n} a_k\right) > 0$, then the nonlocal problem (1)-(3) and the integral equation

$$x(t) = \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_0^r f(s,x_0)\,ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f(s,x(s-r))\,ds\right) + \int_0^r f(s,x_0)\,ds + \int_r^t f(s,x(s-r))\,ds. \tag{4}$$

are equivalent.

Proof. Operating with $I = \int\limits_0^t (.)\,ds$ to both sides of equation (1), we get

$$x(t) = x(0) + \int_0^r f(s,x_0)\,ds + \int_r^t f(s,x(s-r))\,ds. \tag{5}$$

Let $t = t_k$ in equation (5), then we get

$$x(t_k) = x(0) + \int_0^r f(s,x_0)\,ds + \int_r^{t_k} f(s,x(s-r))\,ds,$$

$$\sum_{k=1}^{n} a_k x(t_k) = x(0)\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k \int_0^r f(s,x_0)\,ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f(s,x(s-r))\,ds. \tag{6}$$

Substitute from equation (3) into equation (6) we get,

$$x(0) = x(0)\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k \int_0^r f(s,x_0)\,ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f(s,x(s-r))\,ds,$$

$$x(0) - x(0)\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_k \int_0^r f(s,x_0)\,ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f(s,x(s-r))\,ds$$

and

$$x(0) = \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_0^r f(s,x_0)\,ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f(s,x(s-r))\,ds\right) \tag{7}$$

Substitute from equation (7) into equation (5) we obtain (4).
To complete the proof, differentiating (4) we obtain (1). Also, let $t = 0$ in (4), then by direct calculations we can get (3).

2.2 The solution algorithm

The solution algorithm of equation (4) using ADM is

$$x_0(t) = \left(1 - \sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_0^r f(s, x_0)\, ds \right) + \int_0^r f(s, x_0)\, ds\ , \tag{8}$$

$$x_m(t) = \left(1 - \sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} A_{m-1}(s-r)\, ds \right) + \int_r^{t} A_{m-1}(s-r)\, ds. \tag{9}$$

where A_m are Adomian polynomials of the nonlinear term $f(t, x(t-r))$ which take the form,

$$A_m = \frac{1}{m!} \frac{d^m}{d\lambda^m} \left[f\left(t, \sum_{i=0}^{\infty} \lambda^i x_i(t-r)\right)\right]_{\lambda=0}$$

Finally, the solution of problem (1)-(3) will be

$$x(t) = \sum_{i=0}^{\infty} x_i(t). \tag{10}$$

3 Convergence Analysis

3.1 Existence and Uniqueness theorem

Define the mapping $F : E \to E$ where E is the Banach space $(C(J), \|\cdot\|)$ of all continuous functions on J with the norm $\|x\| = \max_{t \in J} |x(t)|$.

Assume now that the function $f : [0, T] \times R \to R$ is continuous and satisfies the Lipschitz condition

$$|f(t, x(t-r)) - f(t, y(t-r))| \leq k\, |x(t-r) - y(t-r)| \tag{11}$$

Theorem 1: *Let f satisfies the Lipschitz condition (11), then the integral equation (4); which equivalent to problem (1)-(3), has a unique solution $x \in C(J)$.*

Proof: The mapping $F : E \to E$ is defined as,

$$\begin{aligned} Fx &= \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_0^r f(s,x_0)\,ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f(s,x(s-r))\,ds \right) \\ &\quad + \int_0^r f(s,x_0)\,ds + \int_r^t f(s,x(s-r))\,ds \end{aligned}$$

Let $x, y \in E$, then

$$\begin{aligned} Fx - Fy &= \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} [f(s,x(s-r)) - f(s,y(s-r))]\,ds \right) \\ &\quad + \int_r^t [f(s,x(s-r)) - f(s,y(s-r))]\,ds \end{aligned}$$

which implies that

$$\begin{aligned} |Fx - Fy| &= \left| \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} [f(s,x(s-r)) - f(s,y(s-r))]\,ds \right) \right. \\ &\quad \left. + \int_r^t [f(s,x(s-r)) - f(s,y(s-r))]\,ds \right| \\ &\le \left| \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} [f(s,x(s-r)) - f(s,y(s-r))]\,ds \right) \right| \\ &\quad + \left| \int_r^t [f(s,x(s-r)) - f(s,y(s-r))]\,ds \right| \\ &\le \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \sum_{k=1}^{n} a_k \int_r^{t_k} |f(s,x(s-r)) - f(s,y(s-r))|\,ds \\ &\quad + \int_r^t |f(s,x(s-r)) - f(s,y(s-r))|\,ds \\ &\le k \left[\left(1-\sum_{k=1}^{n} a_k\right)^{-1} \sum_{k=1}^{n} a_k \int_r^{t_k} |x(s-r) - y(s-r)|\,ds + \int_r^t |x(s-r) - y(s-r)|\,ds \right] \end{aligned}$$

$$\begin{aligned}\max_{t\in J}|Fx-Fy| &\leq k\left[\left(1-\sum_{k=1}^{n}a_k\right)^{-1}\sum_{k=1}^{n}a_k\max_{t\in J}\int_{r}^{t_k}|x(s-r)-y(s-r)|\,ds\right.\\ &\left.+\max_{t\in J}\int_{r}^{t}|x(s-r)-y(s-r)|\,ds\right]\end{aligned}$$

$$\begin{aligned}\|Fx-Fy\| &\leq k\left[\left(1-\sum_{k=1}^{n}a_k\right)^{-1}\sum_{k=1}^{n}a_k\int_{r}^{t_k}ds+\int_{r}^{t}ds\right]\|x-y\|\\ &\leq k(T-r)\left[\left(1-\sum_{k=1}^{n}a_k\right)^{-1}\left(\sum_{k=1}^{n}a_k\right)+1\right]\|x-y\|\end{aligned}$$

Now, if $k(T-r)\left[\left(1-\sum\limits_{k=1}^{n}a_k\right)^{-1}\left(\sum\limits_{k=1}^{n}a_k\right)+1\right]<1$, then we get

$$\|Fx-Fy\|\leq\|x-y\|,$$

therefore the mapping F is contraction and there exists a unique solution $x\in C(J)$ to the nonlocal Cauchy problem (1)-(3) given by (4), where

$$x(0)=\lim_{t\to 0}x(t)=\left(1-\sum_{k=1}^{n}a_k\right)^{-1}\left(\sum_{k=1}^{n}a_k\int_{0}^{r}f(s,x_0)\,ds+\sum_{k=1}^{n}a_k\int_{r}^{t_k}f(s,x(s-r))\,ds\right)$$

and

$$\begin{aligned}x(T) &= \lim_{t\to T}x(t)=\left(1-\sum_{k=1}^{n}a_k\right)^{-1}\left(\sum_{k=1}^{n}a_k\int_{0}^{r}f(s,x_0)\,ds+\sum_{k=1}^{n}a_k\int_{r}^{t_k}f(s,x(s-r))\,ds\right)\\ &+\int_{0}^{r}f(s,x_0)\,ds+\int_{r}^{T}f(s,x(s-r))\,ds.\end{aligned}$$

This completes the proof. ■

3.2 Proof of convergence

Theorem 2: *The series solution (10) of the problem (1)-(3) using ADM converges if $|x_1(t)|<c, c$ is a positive constant.*

Proof: Define the sequence $\{S_p\}$ such that, $S_p=\sum\limits_{i=0}^{p}x_i(t)$ is the sequence of partial sums from the series solution $\sum_{i=0}^{\infty} x_i(t)$ since,

$$f\left(t, x\left(t-r\right)\right) = \sum_{i=0}^{\infty} A_i,$$

so,

$$f\left(t, S_p\right) = \sum_{i=0}^{p} A_i,$$

From equations (9) and (10) we have,

$$\begin{aligned}\sum_{i=0}^{\infty} x_i &= \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_0^r f\left(s, x_0\right) ds + \sum_{k=1}^{n} a_k \int_r^{t_k} \sum_{i=0}^{\infty} A_{i-1}\left(s-r\right) ds\right) \\ &+ \int_0^r f\left(s, x_0\right) ds + \int_r^t \sum_{i=0}^{\infty} A_{i-1}\left(s-r\right) ds\end{aligned}$$

Let S_p and S_q be two arbitrary partial sums with $p > q$, then we get,

$$\begin{aligned}S_p &= \sum_{i=0}^{p} x_i = \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_0^r f\left(s, x_0\right) ds + \sum_{k=1}^{n} a_k \int_r^{t_k} \sum_{i=1}^{p} A_{i-1}\left(s-r\right) ds\right) \\ &+ \int_0^r f\left(s, x_0\right) ds + \int_r^t \sum_{i=1}^{p} A_{i-1}\left(s-r\right) ds\end{aligned}$$

and

$$\begin{aligned}S_q &= \sum_{i=0}^{q} x_i = \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_0^r f\left(s, x_0\right) ds + \sum_{k=1}^{n} a_k \int_r^{t_k} \sum_{i=1}^{q} A_{i-1}\left(s-r\right) ds\right) \\ &+ \int_0^r f\left(s, x_0\right) ds + \int_r^t \sum_{i=1}^{q} A_{i-1}\left(s-r\right) ds\end{aligned}$$

Now, we are going to prove that $\{S_p\}$ is a Cauchy sequence in this Banach space E.

$$\begin{aligned}S_p - S_q &= \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} \left[\sum_{i=1}^{p} A_{i-1}\left(s\right) - \sum_{i=1}^{q} A_{i-1}\left(s\right)\right] ds\right) \\ &+ \int_r^t \left[\sum_{i=1}^{p} A_{i-1}\left(s\right) - \sum_{i=1}^{q} A_{i-1}\left(s\right)\right] ds\end{aligned}$$

$$\begin{aligned}
&= \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} \left[\sum_{i=q+1}^{p} A_{i-1}\right] ds\right) + \int_r^{t} \left[\sum_{i=q+1}^{p} A_{i-1}\right] ds \\
&= \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} \left[\sum_{i=q}^{p-1} A_{i}\right] ds\right) + \int_r^{t} \left[\sum_{i=q}^{p-1} A_{i}\right] ds \\
&= \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} \left[f\left(t,S_{p-1}\right)-f\left(t,S_{q-1}\right)\right] ds\right) \\
&\quad + \int_r^{t} \left[f\left(t,S_{p-1}\right)-f\left(t,S_{q-1}\right)\right] ds \\
\left|S_p-S_q\right| &= \left|\left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} \left[f\left(t,S_{p-1}\right)-f\left(t,S_{q-1}\right)\right] ds\right)\right. \\
&\quad \left. + \int_r^{t} \left[f\left(t,S_{p-1}\right)-f\left(t,S_{q-1}\right)\right] ds\right| \\
&\leq k\left[\left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} \left|S_{p-1}-S_{q-1}\right| ds\right) + \int_r^{t} \left|S_{p-1}-S_{q-1}\right| ds\right] \\
\left\|S_p-S_q\right\| &\leq k\left(T-r\right)\left[\left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k\right) + 1\right] \left\|S_{p-1}-S_{q-1}\right\| \\
&\leq \beta \left\|S_{p-1}-S_{q-1}\right\|
\end{aligned}$$

Let $p = q + 1$ then,

$$\left\|S_{q+1}-S_q\right\| \leq \beta \left\|S_q-S_{q-1}\right\| \leq \beta^2 \left\|S_{q-1}-S_{q-2}\right\| \leq \cdots \leq \beta^q \left\|S_1-S_0\right\|$$

From the triangle inequality we have,

$$\begin{aligned}
\left\|S_p-S_q\right\| &\leq \left\|S_{q+1}-S_q\right\| + \left\|S_{q+2}-S_{q+1}\right\| + \cdots + \left\|S_p-S_{p-1}\right\| \\
&\leq \left[\beta^q+\beta^{q+1}+\cdots+\beta^{p-1}\right] \left\|S_1-S_0\right\| \\
&\leq \beta^q\left[1+\beta+\cdots+\beta^{p-q-1}\right] \left\|S_1-S_0\right\| \\
&\leq \beta^q\left[\frac{1-\beta^{p-q}}{1-\beta}\right] \left\|x_1\right\|
\end{aligned}$$

Since, $0 < \beta = k\left(T-r\right)\left[\left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k\right) + 1\right] < 1$, and $p > q$ then,

$(1-\beta^{p-q}) \leq 1$. Consequently,

$$\begin{aligned} \|S_p - S_q\| &\leq \frac{\beta^q}{1-\beta}\|x_1\| \\ &\leq \frac{\beta^q}{1-\beta}\max_{t\in J}\ |x_1(t)| \end{aligned}$$

but, $|x_1(t)| < c$ and as $q \to \infty$ then, $\|S_p - S_q\| \to 0$ and hence, $\{S_p\}$ is a Cauchy sequence in this Banach space E so, the series $\sum_{i=0}^{\infty} x_i(t)$ converges. ■

3.3 Error analysis

Theorem 3: *The maximum absolute truncation error of the solution (10) to the problem (1)-(3) is estimated to be,*

$$\left\| x - \sum_{i=0}^{q} x_i \right\| \leq \frac{\beta^q}{1-\beta}\ \|x_1\|$$

Proof: From Theorem 2 we have,

$$\|S_p - S_q\| \leq \frac{\beta^q}{1-\beta}\max_{t\in J}\ |x_1(t)|$$

but, $S_p = \sum_{i=0}^{p} y_i(t)$ as $p \to \infty$ then, $S_p \to y(t)$ so,

$$\|x - S_q\| \leq \frac{\beta^q}{1-\beta}\ \|x_1\|$$

so, the maximum absolute truncation error in the interval J is,

$$\left\| x - \sum_{i=0}^{q} x_i \right\| \leq \frac{\beta^q}{1-\beta}\ \|x_1\|$$

and this completes the proof. ■

4 Numerical Examples

The following examples will be solved by using ADM method and the solution will be compared by using Picard method.

Example 1 Let $\alpha > 0$. Consider the following example,

$$\frac{dx}{dt} = \frac{1}{20}x^2(t-0.1), \qquad t \in (0,10]\,, \tag{12}$$

$$x(t) = 1, \quad t < 0, \tag{13}$$

$$x(0) = \alpha x\left(\frac{1}{2}\right). \tag{14}$$

We prove here, firstly, that as $\alpha \to 0$ the solution of this nonlocal problem continuo to the solution of the usual Cauchy problem (with $\alpha = 0$). This proves the validity of our algorithm.
Using equation (8), problem (12)-(14) will be

$$\begin{aligned} x(t) &= \frac{\alpha}{1-\alpha}\left[\int_0^{0.1} \frac{1}{20} ds + \int_{0.1}^{1/2} \frac{1}{20} x^2 (s-0.1)\, ds\right] \\ &+ \int_0^{0.1} \frac{1}{20} ds + \int_{0.1}^{t} \frac{1}{20} x^2 (s-0.1)\, ds \end{aligned} \tag{15}$$

<u>Solution using ADM method:</u>
Applying ADM to equation (15), we have

$$x_0(t) = \frac{0.005}{1-\alpha}, \tag{16}$$

$$x_i(t) = \frac{\alpha}{20(1-\alpha)} \int_{0.1}^{1/2} A_{i-1}(s-0.1)\, ds + \frac{1}{20}\int_{0.1}^{t} A_{i-1}(s-0.1)\, ds, \; i \geq 1. \tag{17}$$

From equations (16) and (17), the solution of the problem (12)-(14) is,

$$x(t) = \sum_{i=0}^{m} x_i(t). \tag{18}$$

<u>Solution using Picard method:</u>
Applying Picard method to equation (15), we have

$$x_0(t) = \frac{0.005}{1-\alpha}, \tag{19}$$

$$x_i(t) = \frac{0.005}{1-\alpha} + \frac{\alpha}{20(1-\alpha)} \int_{0.1}^{1/2} x_{i-1}^2(s-0.1)\, ds + \frac{1}{20}\int_{0.1}^{t} x_{i-1}^2(s-0.1)\, ds, \; i \geq 1. \tag{20}$$

the solution of the problem (12)-(14) using Picard method will be,

$$x(t) = x_m(t). \tag{21}$$

Figures 1.a - 1.d show a comparison between ADM and Picard solutions (when $\alpha = 0.1, 0.001, 0.00001, 0$ respectively, and $m = 5$).

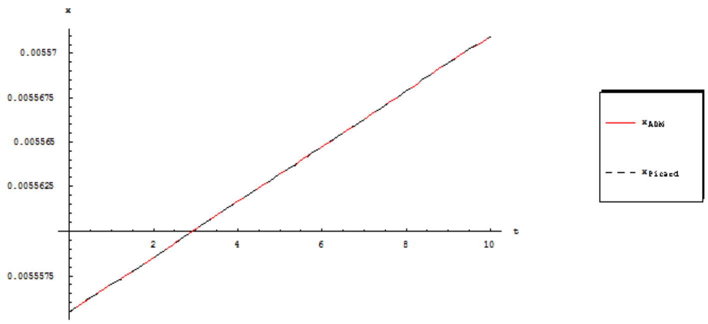

Fig (1-a): ADM and Picard solutions [$\alpha = 0.1$]

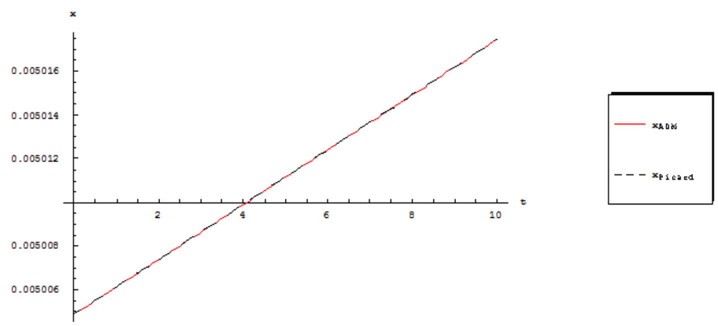

Fig (1-b): ADM and Picard solutions [$\alpha = 0.001$]

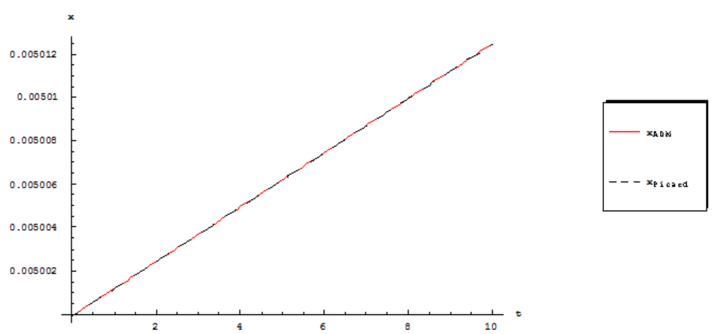

Fig (1-c): ADM and Picard solutions [$\alpha = 0.00001$]

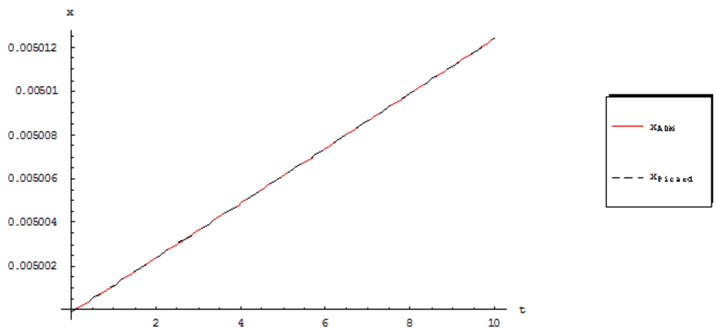

Fig (1-d): ADM and Picard solutions $[\alpha = 0]$

Table (1.a) shows the absolute error between ADM solution and Picard solution (when $m = 5, \alpha = 0.1$).

Table (1.a)

t	*Absolute error*
1	4.2304×10^{-19}
2	4.2369×10^{-19}
3	4.25117×10^{-19}
4	4.30775×10^{-19}
5	4.50657×10^{-19}
6	5.06871×10^{-19}
7	6.40971×10^{-19}
8	9.23075×10^{-19}
9	1.46274×10^{-18}
10	2.42164×10^{-18}

Table (1.b) shows a comarasion between the time of ADM solution and Picard solution (when $m = 5, \alpha = 0.1$).

Table (1.b)

ADM time	*Picard time*
0.155 sec.	0.234 sec.

Example 2 *Consider the following nonlocal DE,*

$$\frac{dx}{dt} = \frac{1}{10} t^2 e^{x^2(t-0.5)}, \qquad t \in (0, 4], \tag{22}$$

$$x(t) = \frac{1}{2}, \quad t < 0, \tag{23}$$

$$x(0) = \frac{1}{2} x\,(0.7) - \frac{1}{4} x\,(0.9)\,. \tag{24}$$

Using equation (8), problem (22)-(24) will be

$$x(t) = \frac{e^{1/4}}{30}\int_0^{0.5} s^2 ds + \frac{1}{15}\int_{0.5}^{0.7}\left[s^2 e^{x^2(s-0.5)}\right] ds - \frac{1}{30}\int_{0.5}^{0.9}\left[s^2 e^{x^2(s-0.5)}\right] ds$$
$$+\frac{e^{1/4}}{10}\int_0^{0.5} s^2 ds + \frac{1}{10}\int_{0.5}^{t} s^2 e^{x^2(s-0.5)} ds, \tag{25}$$

<u>**Solution using ADM method:**</u>
Applying ADM to equation (25), we have

$$x_0(t) = \frac{e^{1/4}}{30}\int_0^{0.5} s^2 ds + \frac{e^{1/4}}{10}\int_0^{0.5} s^2 ds, \tag{26}$$
$$x_i(t) = \frac{1}{15}\int_{0.5}^{0.7} s^2 A_{i-1}(s-0.5)\, ds - \frac{1}{30}\int_{0.5}^{0.9} s^2 A_{i-1}(s-0.5)\, ds$$
$$+\frac{1}{10}\int_{0.5}^{t} s^2 A_{i-1}(s-0.5)\, ds, \qquad i \geq 1. \tag{27}$$

From equations (26) and (27), the solution of the problem (22)-(24) is,

$$x(t) = \sum_{i=0}^{m} x_i(t). \tag{28}$$

<u>**Solution using Picard method:**</u>
Applying Picard method to equation (15), we have

$$x_0(t) = \frac{e^{1/4}}{30}\int_0^{0.5} s^2 ds + \frac{e^{1/4}}{10}\int_0^{0.5} s^2 ds, \tag{29}$$
$$x_i(t) = \frac{e^{1/4}}{30}\int_0^{0.5} s^2 ds + \frac{e^{1/4}}{10}\int_0^{0.5} s^2 ds + \frac{1}{15}\int_{0.5}^{0.7} s^2 e^{x_{i-1}^2(s-0.5)} ds$$
$$-\frac{1}{30}\int_{0.5}^{0.9} s^2 e^{x_{i-1}^2(s-0.5)} ds + \frac{1}{10}\int_{0.5}^{t} s^2 e^{x_{i-1}^2(s-0.5)} ds, \quad i \geq 1. \tag{30}$$

the solution of the problem (22)-(24) using Picard method will be,

$$x(t) = x_m(t). \tag{31}$$

Figure 2 shows a comparison between ADM solution (when $m = 5$) and Picard solution (when $m = 2$).

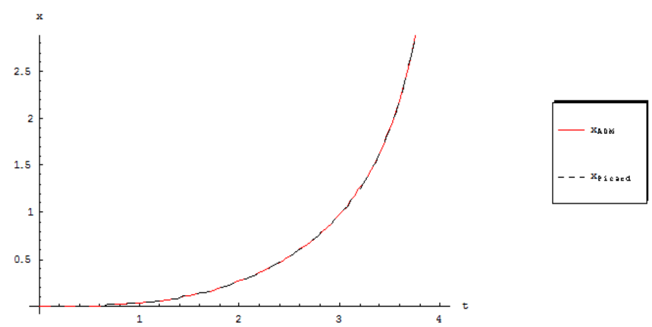

Fig (2): ADM and Picard solutions.

Table (2.a) shows the absolute error between ADM solution (when $m = 5$) and Picard solution (when $m = 2$).

Table (2.a)

t	*Absolute error*
0.5	2.57408×10^{-12}
1	2.43143×10^{-11}
1.5	2.11219×10^{-9}
2	1.65101×10^{-7}
2.5	0.000032022
3	0.0011126
3.5	0.0105128
4	0.262885

Table (2.b) shows a comarasion between the time of ADM solution (when $m = 5$) and Picard solution (when $m = 2$).

Table (2.b)

ADM time	*Picard time*
0.296 sec.	5.693 sec.

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