

## Study of $f(T)$ dark energy model using different choices of the cosmological scale factor

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### Abstract

In the framework of the teleparallel gravity or the  $f(T)$  theory, we have investigated the behaviors of some cosmological parameters as a function of redshift  $z$  using several dark energy models in intermediate, logamediate and emergent scenarios of the universe adopting their cosmological scale factors. The results show that the universe has an accelerated expansion behavior described by phantom-like behavior. The stability of the model has been studied using the squared speed of sound  $v_s^2$  verifying that the model is stable. Finally, we discussed the generalized second law of thermodynamics's (GSLT) validity. The positive behavior of the entropy indicates that the GSLT is verified.

*Keywords:* Dark Energy; modified gravity; scale factor; accelerated behavior; inflation.

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## 1 Introduction

The universe has an expanded acceleration behavior. The observational data have been obtained various cosmological experiments of the type Ia supernovae (SNeIa), large scale structure (LSS), X-ray experiments, the Wilkinson Microwave Anisotropy Probe (WMAP), the Planck satellite, the Sloan Digital Sky Survey (SDSS) and cosmic microwave background (CMB) anisotropy radiation, which indicated that our universe has an accelerated expansion behavior [Ghosh, R et al. (2012); Pasqua, A et al. (2016, 2014)]. The accelerated expansion represents the present day problems in cosmology. Mainly, a few classes of models exist to explain the accelerated expansion nature of the universe at the present time [Chattopadhyay, S and et al. (2014); Jawad, A et al. (2017); Pasqua, A et al. (2014)].

Cosmological constant model represents the simplest explanation for the accelerated behavior of the universe and has the Equation of State (EoS) parameter ( $\omega_\Lambda = -1$ ), which should be assumed to clarify the theoretical candidate suggested to explain the observational evidence of accelerated expansion of the universe. We face two main problems described as follows: the fine-tuning and the

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cosmic coincidence problems [Chattopadhyay, S and et al. (2014); Pasqua, A et al. (2014)].

Some theoretical models considered dark energy (DE) as natural dynamic like [Tajahmad, B. (2018)] quintessence ( $-1 \leq \omega_\Lambda \leq 1$ ) [Nojiri, S and Odintsov, SD. (2007); Copeland, EJ et al. (2006)], phantom ( $\omega_\Lambda \leq -1$ ) [Copeland, EJ et al. (2006)], Chaplygin gas ( $\omega_\Lambda < -1/3$ ) [Bilic, N et al. (2002)] and k-essence ( $\omega_\Lambda < -1/3$ ) [Copeland, EJ et al. (2006)]. In fact, some models of dark energy based on the holographic principle [Ghosh, R et al. (2012)] that is described by two different types of holographic [Chattopadhyay, S and et al. (2014); Pasqua, A et al. (2014, 2016)] and agegraphic dark energy [Karami, K et al. (2010)]. Another interesting suggestions to DE mysterious is the modified gravity which was proposed after the failure of general relativity (GR). This new set of gravity theories passes many tests of solar system and astrophysical successfully [Jamil, M. (2012)]. The modified Gravity theories are important for the late accelerated expansion of the universe which has been studied before by many authors [Chattopadhyay, S and et al. (2014)]. Some of the familiar models of the modified gravity are  $f(T)$ ,  $f(R)$  and  $f(G)$ , where  $T$  is the torsion scalar,  $R$  is the Ricci scalar curvature and  $G$  is the Gauss-Bonnet invariant [Jawad, A et al. (2017); Pasqua, A et al. (2014); Yousaf, Z. (2019); Del Popolo, A et al. (2019)].

In the present work, we investigate the modified  $f(T)$  gravity models of dark energy using three different scale factors. We determine some cosmological parameters to explain the acceleration behavior of the universe. Finally, we compare our results with present observations obtained at varying time points. The paper is organized in the following sections. In section 2, we study the modified gravity model of  $f(T)$ . In section 3, we define three various scenarios of the scale factors. We investigate the accelerated expansion behavior of the universe using the cosmological parameters are defined in section 4, the EoS parameter is represented in section 5, checking the stability of model in section 6 and the validity of the GSLT, which is presented in section 7. Finally, The results are discussed in section 8.

## 2 The model

Not necessary constructing a gravitational theory on a Riemannian manifold. A manifold can be divided into two separated parts but connected; one with Riemannian structure with a specific metric and another part with a non-Riemannian structure and with torsion. That part which has the zero Riemannian tensor but has non zero torsion is based on a tetrad basis, and defines a Weitzenbock space-time. The  $f(T)$  modified gravity is an alternative theory for GR, defined on the Weitzenbock non-Riemannian manifold, working only with torsion. This model is constructed by Einstein for unifying the electromagnetism and gravity [Jamil, M. (2012)]. If we put  $f(T) = T$ , this theory is called the teleparallel gravity [Daouda, MH et al. (2012); Li, B et al. (2011)].

In this work, the  $f(T)$  is a general form of the teleparallel gravity (TG). In the framework of  $f(T)$  theory, the action  $S$  is represented by [Karami, K and Abdolmaleki, A. (2012)]:

$$S = \frac{1}{16\pi G_N} \int \sqrt{-g} d^4x (f(T) + L_m), \quad (2.1)$$

where  $G_N$  indicates the Newton's gravitational constant,  $g$  is the determinant of the metric tensor,  $f(T)$  is general differentiable function of the torsion and  $L_m$  is the Lagrangian density of the matter inside the universe.

According to Eq.(2.1), the action  $S$  of a homogeneous and isotropic universe is described from the geometrical point of view, the Friedmann-Lemaître-Robertson-Walker (FLRW) model is represented

by the Friedman-Robertson-Walker (FRW) metric [Arabsalmani, M and Sahni, V. (2011)]:

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \right), \quad (2.2)$$

where  $a(t)$  represents the scale factor,  $r$  indicates the radial component of FRW metric and  $t$  is the cosmic time. Furthermore, the term of  $r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$  is the solid angle element squared and the quantities  $\theta$  and  $\varphi$  are the usual azimuthal and polar angles that are assumed in the range of  $0 \leq \theta \leq \pi$  and  $0 \leq \varphi \leq 2\pi$  [Pasqua, A et al. (2016)].

The expansion factor of the universe can be assumed as a Taylor series at the present time  $t_o$  [Arabsalmani, M and Sahni, V. (2011)]:

$$a(t) = a_o \left( 1 + \sum_{n=1}^{\infty} \frac{A_n(t_o)}{n!} (H_o(t - t_o))^n \right), \quad (2.3)$$

where  $A_n = \frac{a^{(n)}}{a H^n}$ ,  $n \in N$  and  $a^{(n)}$  is the  $n^{th}$  derivative of the scale factor with respect to the time scale [Ghosh, R et al. (2012); Arabsalmani, M and Sahni, V. (2011)].

In this work, the  $f(T)$  form, including constant, linear and non-linear of torsion  $T$ , is given by [Jamil, M. (2012); Jawad, A et al. (2017)],

$$f(T) = T + \frac{6(n-1)^u}{u-1} \left( -\frac{T}{6n^2} \right)^{u/2} + \frac{\sqrt{-T}}{\sqrt{6n}}, \quad (2.4)$$

where  $u, n$  are constants and  $T$  is the torsion scalar [Pasqua, A et al. (2014)]. The Hubble parameter is  $H = \frac{\dot{a}}{a}$  and the torsion scalar is assigned to [Jawad, A et al. (2017)]:

$$T(t) = -6H^2 = -\frac{6n^2}{t^2}. \quad (2.5)$$

The Hubble horizon is taken as the infrared (IR) cut-off, while the holographic dark energy density is found to be [Pasqua, A et al. (2014)]:

$$\rho_{\Lambda} = 3c^2 M_p^2 L^{-2}, \quad (2.6)$$

where  $M_p = (8\pi G_N)^{-\frac{1}{2}} \approx 10^{18} \text{ GeV}$  indicates the reduced Planck mass and  $L$  is the IR cut-off size [Xu, L et al. (2009)]. Gao et al. [Gao, C et al. (2009)] took the Ricci scalar as an IR cut-off, where the Ricci dark energy is proportional to the Ricci scalar [Xu, L et al. (2009)]. The Ricci scalar curvature of FLRW universe is appointed [Pasqua, A et al. (2014)]:

$$R = -6 \left( \dot{H} + 2H^2 + \frac{k}{a^2} \right), \quad (2.7)$$

where  $k$  is the curvature parameter, which can be  $\mp 1$  and  $0$ , respectively, yielding an open, a closed, or a flat FRW universe. Let us consider DE component is proportional to the inverse of squared Ricci scalar curvature radius as described in the holographic principle term [Gao, C et al. (2009)], where energy density  $\rho$  of Ricci dark energy (RDE) model for a flat FLRW universe is given by, for  $k = 0$  [Pasqua, A et al. (2014)],

$$\rho = 3c^2 \left( \dot{H} + 2H^2 \right). \quad (2.8)$$

By differentiating Eq.(2.8), the time derivative of dark energy density is described by [Pasqua, A et al. (2014)]:

$$\dot{\rho} = 3c^2 \left( \ddot{H} + 4\dot{H}H \right), \quad (2.9)$$

where  $c^2$  is a positive constant.

### 3 Different Scenarios of the Scale Factors

#### 3.1 Intermediate Inflation

The expansion of universe is expressed in exponential form. The intermediate expansion scale factor is defined as [Pasqua, A et al. (2014)]:

$$a = e^{Bt(z)^n}, \quad (3.1)$$

where  $B > 0$  and  $0 < n < 1$ .

#### 3.2 Logamediate Inflation

We would study another generalized form of inflation, which call 'logamediate inflation'. The scale factor has the form [Barrow, JD and Nunes, N. (2007)]:

$$a = e^{A(\text{Log}t(z))^n}. \quad (3.2)$$

where  $A > 0$  and  $n > 1$ .

We would clarify some calculations to the behavior of higher-order characteristics for the logamediate inflation.

#### 3.3 Emergent Inflation

The Emergent Inflation presents a framework to discuss the suggestions if whatever process caused the universe to come into being, preferred the high-symmetry state of the Einstein static model to any less ordered state [Ellis, GF and Maartens, R. (2003)]. For the emergent universe, the scale factor can be chosen as [Debnath, U. (2013)]:

$$a = a_o \left( \lambda + e^{\mu t(z)} \right)^n, \quad (3.3)$$

where  $a_o$ ,  $\lambda$ ,  $\mu$  and  $n$  are positive constants [Pasqua, A et al. (2014); Debnath, U. (2013)],

1.  $a_o > 0$  for the scale factor  $a$  to be positive.
2.  $a > 0$  or  $n > 0$  for expanding model of the universe.
3.  $a < 0$  and  $n < 0$  imply big bang singularity at  $t = -\infty$ .
4.  $\lambda > 0$  to avoid any uniqueness at finite time (big rip).

In the next sections, we are going to establish the expressions of some cosmological parameters that is necessary to describe the behavior of the considered different scenarios of scale factors.

### 4 Cosmological parameters

The higher-order characteristics of the universe could be related to the accelerated expansion behavior. The Hubble parameter is derived from [Ghosh, R et al. (2012); Dkabrowski, MP (2005)]:

$$H = \frac{\dot{a}}{a}. \quad (4.1)$$

The deceleration parameter is given by [Ghosh, R et al. (2012)]:

$$q = -\frac{1}{H^2} \frac{\ddot{a}}{a}. \quad (4.2)$$

The jerk parameter is [Ghosh, R et al. (2012)]:

$$j = \frac{1}{H^3} \frac{\overset{\dots}{a}}{a}. \quad (4.3)$$

The kerk or snap parameter is [Ghosh, R et al. (2012)]:

$$\kappa = -\frac{a^4}{aH^4}. \quad (4.4)$$

The lerk parameter is [Ghosh, R et al. (2012)]:

$$l = \frac{1}{H^5} \frac{a^{[5]}}{a}. \quad (4.5)$$

The general form of these parameters can be expressed by [Dkabrowski, MP (2005)]:

$$x^i = \frac{(-1)^{i+1} a^i}{aH^i} = (-1)^{i+1} \frac{a^i a^{i-1}}{\overset{i}{a}^{i+1}}. \quad (4.6)$$

where  $i$ -th indicates the order of the derivative with respect to the cosmic time  $t$  and  $i$  is just a power. Eq.(4.2) is the universal acceleration can be quantified through a dimensionless cosmological function. For  $q < 0$ , it describes the accelerated universe. While for  $q \geq 0$ , it is simplified for either decelerated or expanded universe at  $a \propto t$ . The dimensionless third time derivative of the scale factor with respect to cosmic time is defined in Eq.(4.3), it is known as "jerk parameter" and also "statefinder parameter". Eq.(4.4) has the fourth time derivative of scale factor. While Eq.(4.5) is the fifth time derivative of scale factor [Dkabrowski, MP (2005); Ghosh, R et al. (2012)].

#### Intermediate Inflation:

Let appoint some calculations for the intermediate inflation, it is useful to write down the higher-order time derivatives of the scale factor of the intermediate expansion, see Eq.(3.1), then

$$\dot{a} = Bnt(z)^{n-1} e^{Bt(z)^n}. \quad (4.7)$$

$$\ddot{a} = Bnt(z)^{n-1} e^{Bt(z)^n}. \quad (4.8)$$

$$\overset{\dots}{a} = Bnt(z)^{n-2} e^{Bt(z)^n} (Bnt(z)^n + n - 1). \quad (4.9)$$

$$a^{(4)} = Bnt(z)^{n-3} e^{Bt(z)^n} \times (B^2 n^2 t(z)^{2n} + 3B(n-1)nt(z)^n + n^2 - 3n + 2). \quad (4.10)$$

$$a^{(5)} = Bnt(z)^{n-4} e^{Bt(z)^n} (B^3 n^3 t(z)^{3n} + 6B^2(n-1)n^2 t(z)^{2n} + Bn(7n^2 - 18n + 11) t(z)^n + n^3 - 6n^2 + 11n - 6). \quad (4.11)$$

It's well known that the relation between cosmic time and redshift  $z$  is [Macdonald, A. (2006)]:

$$t(z) = \frac{2}{((z+1)^2 + 1) H_o}. \quad (4.12)$$

The present time  $t_o$  at  $z = 0$  can be obtained from the relation:

$$t_o = \frac{1}{H_o}. \quad (4.13)$$

Substitute the present value of Hubble parameter  $H_o \simeq 74$  in Eq.(4.13), we get:

$$t_o \approx 0.0135. \tag{4.14}$$

Adopting Eq.(4.2), Eq.(4.3), Eq.(4.4) and Eq.(4.5) for the intermediate inflation, we obtain these behaviors as shown in Fig. 1:

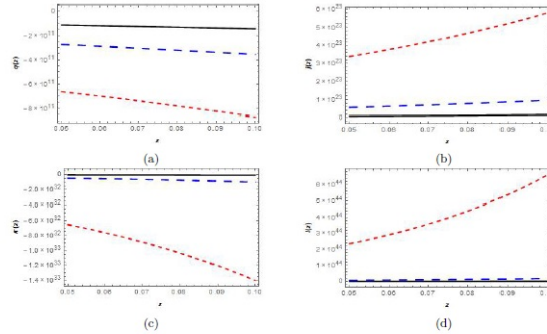


Figure 1: The behavior of (a) deceleration, (b) jerk, (c) kerk and (d) lerk parameters as a function of red-shift  $z$  for the intermediate inflation, for  $n = 5$  (black-solid),  $n = 5.2$  (blue-dashed) and  $n = 5.4$  (red-dashed).

In Fig. 1, we represent the evolution of  $q, j, \kappa$  and  $l$  for different values of  $n$ . In Fig. a, we notice that the deceleration parameter ( $q$ ) has a slight decreasing behavior at negative level ( $q < 0$ ) for all  $n$  values. Consequently, this result indicates that our universe is in continuous accelerating expansion.

Fig. b shows that the jerk parameter ( $j$ ) has a positive increasing behavior against the red-shift  $z$  for three values of  $n$ . For  $n = 5$ , the jerk parameter is  $j \simeq 4.17349 \times 10^{19}$  at  $z \sim 0.05$  and  $j \simeq 6.53741 \times 10^{19}$  at  $z \sim 0.1$ , showing a slow increasing behavior.

Fig. c displays the kerk parameter ( $\kappa$ ) which has a negative decreasing behavior for all values of  $n$ . For  $n = 5$ , the black-solid line from  $\kappa_{z=0.05} \simeq -3.56705 \times 10^{30}$  to  $\kappa_{z=0.1} \simeq -7.16964 \times 10^{30}$  tends to be a constant rate of deceleration. Adopting  $n = 5.2$  (blue-dashed) and  $n = 5.4$  (red-dashed), we notice a decreasing behavior for  $\kappa < 0$ .

As evident from Fig. d that the lerk parameter ( $l$ ) has an increasing behavior at positive level. For  $n = 5$ , we find the black-solid line from  $l_{z=0.05} \simeq 2.03249 \times 10^{41}$  to  $l_{z=0.1} \simeq 5.242 \times 10^{41}$  tends to be zero. For  $n = 5.2$  (blue-dashed) and  $n = 5.4$  (red-dashed), the behavior is found to be increased for  $l > 0$ .

**Logamediate inflation:**

Using the scale factor of the logamediate inflation given in Eq.(3.2), the time derivatives are explicated as:

$$\dot{a} = \frac{An}{t(z)} \log(t(z))^{n-1} e^{A \log(t(z))^n}. \tag{4.15}$$

$$\ddot{a} = \frac{An \log(t(z))^{n-1} e^{A \log^n(t(z))}}{t(z)}. \tag{4.16}$$

$$\dots = \frac{An \log(t(z))^{n-2} e^{A \log^n(t(z))}}{t(z)^2} \times (An \log(t(z))^n - \log(t(z)) + n - 1). \tag{4.17}$$

$$a^{(4)} = \frac{An \log(t(z))^{n-3} e^{A \log(t(z))^n}}{t(z)^3} \times (A^2 n^2 \log(t(z))^{2n} - 3An \log(t(z))^{n+1} + 3A(n-1)n \log(t(z))^n - 3(n-1) \log(t(z))^2 + n^2 - 3n + 2), \tag{4.18}$$

$$\begin{aligned}
 a^{(5)} &= \frac{An \log(t(z))^{n-4} e^{A \log^n(t(z))}}{t(z)^4} \\
 &\times (A^3 n^3 \log(t(z))^{3n} + 6A^2 n^2 (n-1) \log^{2n}(t(z)) \\
 &- 6A^2 n^2 \log(t(z))^{2n+1} + An(7n^2 - 18n + 11) \log(t(z))^n \\
 &- 18An(n-1) \log(t(z))^{n+1} + 11An \log(t(z))^{n+2} \\
 &- 6 \log(t(z))(n^2 - 3n + 2) + 11(n-1) \log(t(z))^2 \\
 &- 6 \log(t(z))^3 + n^3 - 6n^2 + 11n - 6),
 \end{aligned} \tag{4.19}$$

The behaviors of the cosmological parameters are presented in Fig. 2:

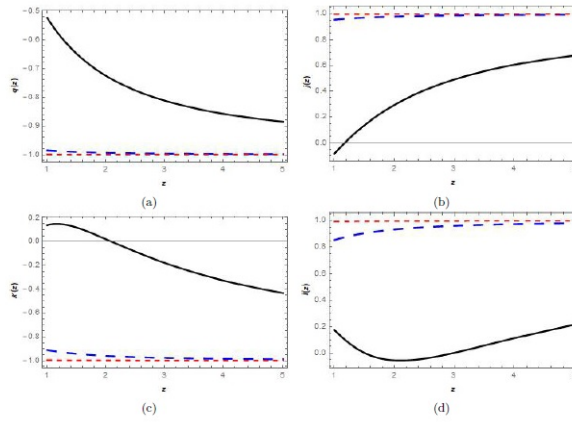


Figure 2: The behavior of (a) deceleration, (b) jerk, (c) kerk and (d) lerk parameters as a function of red-shift  $z$  for logamediate inflation when  $n = 5$  (black-solid),  $n = 7$  (blue-dashed) and  $n = 9$  (red-dashed).

The parameters  $q$ ,  $j$ ,  $\kappa$  and  $l$  that expressed in Fig. 2 versus the red-shift  $z$ . We notice from Fig. a that a decreasing behavior at negative level as obtained from SNela data for all  $n$  values. The negative level of  $q < 0$  indicates the universe has an accelerated behavior and from  $q_{z=1} \simeq -0.524361$  to  $q_{z=3} \simeq -0.812034$  show a slowly decreased behavior.

Fig. b has an increasing behavior ( $j > 0$ ) for three values of  $n$ . Values of  $n = 7$  (blue-dashed) and  $n = 9$  (red-dashed) show a tiny increased behavior and tend to be constant.

In Fig. c, we obtain a decreasing behavior remains negative ( $-1 < \kappa < 0$ ) for all values of  $n$ .

Fig. d displays an increasing behavior at positive level for the values of  $n = 7$  (blue-dashed) and  $n = 9$  (red-dashed), i.e.,  $l > 0$ . For the value of  $n = 5$  (black-solid), the behavior decreases from  $l_{z=1} \simeq -0.000199579$  to  $l_{z=2} \simeq -8.85769 \times 10^{-6}$  and begins in increasing at  $l_{z \simeq 2.1} \simeq 5.09981 \times 10^{-7}$ .

**Emergent inflation:**

Doing some algebraic calculations to study the behavior of higher-order characteristics for the emergent inflation. Starting from Eq.(3.3).

$$\dot{a} = \mu n a_o e^{\mu t(z)} \left( \lambda + e^{\mu t(z)} \right)^{n-1}. \tag{4.20}$$

$$\ddot{a} = \mu n a_o e^{\mu t(z)} \left( \lambda + e^{\mu t(z)} \right)^{n-1}. \tag{4.21}$$

$$\ddot{\ddot{a}} = \mu^2 n a_o e^{\mu t(z)} \left( \lambda + e^{\mu t(z)} \right)^{n-2} \left( \lambda + n e^{\mu t(z)} \right). \tag{4.22}$$

$$a^{(4)} = \mu^3 n a_o e^{\mu t(z)} \left( \lambda + e^{\mu t(z)} \right)^{n-3} \left( \lambda^2 + n^2 e^{2\mu t(z)} + \lambda(3n-1)e^{\mu t(z)} \right). \quad (4.23)$$

$$a^{(5)} = \mu^4 n a_o e^{\mu t(z)} \left( \lambda + e^{\mu t(z)} \right)^{n-4} \times \left( \lambda^3 + n^3 e^{3\mu t(z)} + \lambda(6n^2 - 4n + 1)e^{2\mu t(z)} + \lambda^2(7n - 4)e^{\mu t(z)} \right). \quad (4.24)$$

The cosmological parameter in Fig. 3 has the following behaviors:

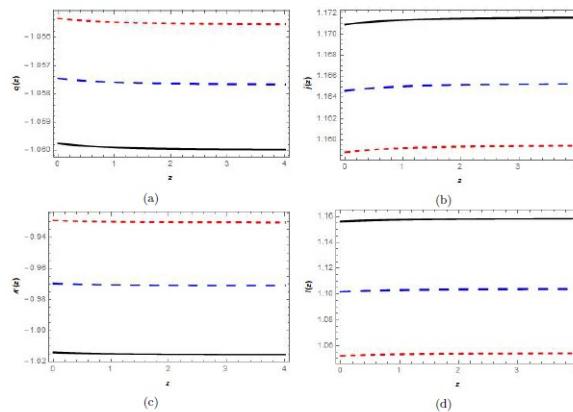


Figure 3: The behavior of (a) deceleration, (b) jerk, (c) kerk and (d) lerk parameters as a function of  $z$  for the emergent inflation, for  $n = 5$  (black-solid),  $n = 5.2$  (blue-dashed) and  $n = 5.4$  (red-dashed).

The evolution of  $q$ ,  $j$ ,  $\kappa$  and  $l$  parameters as a functions of red-shift  $z$  is shown in Fig. 3. From Fig. a, we notice a very small decreasing behavior where  $q < 0$ , indicating a continuous accelerating behavior of the universe.

For the jerk parameter in Fig. b, we also have an increasing behavior for three values of  $n$ .

Fig. c presents a negative decreasing behavior, where  $\kappa < -1$  for all positive values of  $n$ .

The behavior of  $l$  parameter as a function of redshift  $z$  in Fig. d is increasing at positive level, i.e.,  $l > +1$  for different values of  $n$ , from  $z = 0$  to  $z = 4$ .

## 5 The Equation of State parameter

We derive the expression of EoS parameter for the non-interacting and the interacting DE. We start considering the non-interacting case.

The continuity equations are given by [Pasqua, A et al. (2014)]:

$$\dot{\rho}_{\text{tot}} + 3H\rho_{\text{tot}}(\omega_{\text{tot}} + 1) = 0, \quad (5.1)$$

where  $\omega_{\text{tot}}$  is the total EoS parameter. If there is no interaction between DE sectors, the two energy densities  $\rho$  and  $\rho_m$  of DE and dark matter (DM) are derived as [Pasqua, A et al. (2014)]:

$$\dot{\rho} + 3H\rho(\omega_{\Lambda} + 1) = 0, \quad (5.2)$$

$$\dot{\rho}_m + 3H\rho_m = 0, \quad (5.3)$$



where  $\omega_\Lambda$  is the EoS parameter of DE and the EoS parameter of DM is  $\omega_{m=0}$  due to  $p_m = 0$ . Inserting Eq.(2.8) and Eq.(2.9) in Eq.(5.2), we obtain the non-interacting EoS parameter  $\omega_{\Lambda non}$ :

$$\omega_{\Lambda non} = -\frac{\left(\ddot{H} + 4\dot{H}H\right)}{3H\left(\dot{H} + 2H^2\right)} - 1. \tag{5.4}$$

For the interacting case, the continuity equations are expressed as [Pasqua, A et al. (2014)]:

$$\dot{\rho} + 3H\rho(\omega_\Lambda + 1) = -Q, \tag{5.5}$$

$$\dot{\rho}_m + 3H\rho_m = Q, \tag{5.6}$$

where  $Q$  is an arbitrary function of cosmological parameters, called the interaction term and is chosen here as  $Q = 3\delta H\rho_m$  [Pasqua, A et al. (2016, 2014)], where  $\delta$  is the dimensionless coupling between DE and DM and is known as a transfer strength or an interaction parameter. The cosmological observational data obtained from different cosmological experiments determined the coupling parameter between DM and DE, that must be assumed to be small positive value and be in agreement with the experiments. Observations of CMB radiation suggest that the interaction term is  $0 < \delta < 0.025$ . This result is consistent with the interaction term  $\delta$  must be taken in the range of  $(0, 1)$  in the limiting case of  $\delta = 0$  which leads to the non-interacting model [Pasqua, A et al. (2016)].

By solving the continuity equation for  $\rho_m$ , see Eq.(5.6), the energy density of DM is defined as [Pasqua, A et al. (2014)]:

$$\rho_m = \rho_{m0} a^{-3(1+\delta)}, \tag{5.7}$$

where  $\rho_{m0}$  indicates the present time value of  $\rho_m$ .

Adopting Eq.(2.8) and Eq.(2.9) in Eq.(5.5), the interacting EoS parameter  $\omega_{\Lambda inter}$  is expressed as follows:

$$\omega_{\Lambda inter} = -\frac{\delta\rho_m}{\rho} - \frac{\left(\ddot{H} + 4\dot{H}H\right)}{3H\left(\dot{H} + 2H^2\right)} - 1. \tag{5.8}$$

**Intermediate inflation:**

By inserting Eq.(3.1) and Eq.(4.7) together with Eq.(4.1), we get:

$$H = Bnt(z)^{n-1}. \tag{5.9}$$

The First time derivative of Hubble parameter is given as:

$$\dot{H} = Bn(n-1)t(z)^{n-2}. \tag{5.10}$$

By differentiating Eq.(5.10) with respect to cosmic time  $t(z)$ , we obtain:

$$\ddot{H} = Bn(n-1)(n-2)t(z)^{n-3}. \tag{5.11}$$

The behavior of EoS parameter for both cases of non-interacting and interacting DE for the intermediate inflation is expressed in Fig. 4:

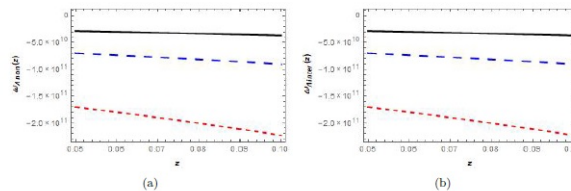


Figure 4: The behavior of EoS parameter as a function of red-shift  $z$  for the (a) non-interacting and (b) interacting intermediate inflation, for  $n = 5$  (black-solid),  $n = 5.2$  (blue-dashed) and  $n = 5.4$  (red-dashed).

Fig. 4 presents the variation of EoS parameter for non-interacting and interacting cases versus redshift  $z$ . It is seen from Fig. a and Fig. b that the EoS parameter is in a decreasing negative behavior which indicates a phantom-like behavior for different values of  $n$ , over the considered range of redshift.

**Logamediate inflation:**

By inserting Eq.(3.2) with Eq.(4.15) in Eq.(4.1), we can obtain:

$$H = \frac{An}{t(z)} \log(t(z))^{n-1}. \tag{5.12}$$

The First derivative of Hubble parameter is afforded as:

$$\dot{H} = \frac{An}{t(z)^2} (\log(t(z))^{n-2}) ((n-1) - \log(t(z))). \tag{5.13}$$

By differentiating Eq.(5.13) with respect to cosmic time  $t(z)$ , we get:

$$\ddot{H} = \frac{An}{t(z)^3} (\log(t(z))^{n-3}) (-3(n-1) \log(t(z)) + (n-2)(n-1) + 2 \log(t(z))^2). \tag{5.14}$$

In Fig. 5, the behaviors of non-interacting and interacting EoS parameter of the logamediate inflation are considered as:

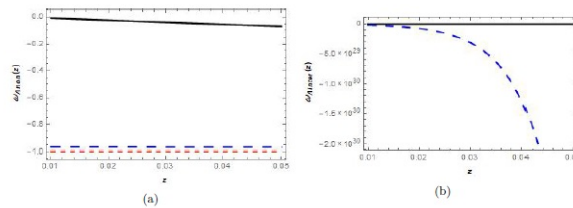


Figure 5: The behavior of EoS parameter as a function of red-shift  $z$  for the (a) non-interacting (b) interacting logamediate inflation for  $n = 5$  (black-solid),  $n = 7$  (blue-dashed) and  $n = 9$  (red-dashed).

Fig. a has a decreasing behavior at negative level  $\omega_{\Lambda non} > -1$ , i.e., the non-interacting case is in a phantom-like behavior, for  $n$  values,  $n = 5$  (black-solid),  $n = 7$  (blue-dashed) and  $n = 9$  (red-dashed). Fig. b represents a negative decreasing behavior for  $n = 7$  where the blue-dashed line for  $\omega_{\Lambda inter, z=0.01} \simeq -0.947018$  indicates a phantom-like behavior. For  $n = 5$ , the black-solid line from  $\omega_{\Lambda inter, z=0.01} \simeq 0.112782$  to  $\omega_{\Lambda inter, z=0.05} \simeq 0.0677319$  shows a quintessence phase, from  $z = 0.01$  to  $z = 0.05$ .

**Emergent inflation:**

From Eq. (3.3) with Eq.(4.20) inserting in Eq.(4.1), the Hubble parameter of the emergent inflation is:

$$H = \frac{n\mu e^{\mu t(z)}}{\lambda + e^{\mu t(z)}}. \tag{5.15}$$

The First time derivative of Hubble parameter is:

$$\dot{H} = \frac{\mu^2 n \lambda e^{\mu t(z)}}{(\lambda + e^{\mu t(z)})^2}. \tag{5.16}$$

By differentiating Eq.(5.16) with respect to cosmic time  $t(z)$ , the result is:

$$\ddot{H} = \frac{\lambda^2 \mu^3 n e^{\mu t(z)} - \mu^3 n \lambda e^{2\mu t(z)}}{(\lambda + e^{\mu t(z)})^3}. \tag{5.17}$$

The behaviors of the EoS parameter for both non-interacting and interacting emergent inflation are expressed in fig. 6:

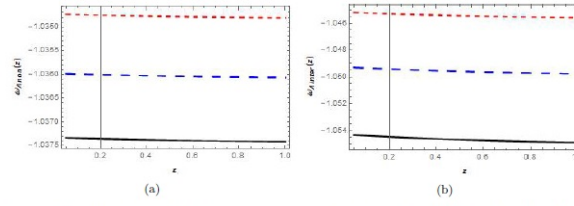


Figure 6: The behavior of EoS parameter as a function of red-shift  $z$  for the (a) non-interacting (b) interacting emergent inflation for  $n = 5$  (black-solid),  $n = 5.2$  (blue-dashed) and  $n = 5.4$  (red-dashed).

In Fig. a, the EoS parameter is a function of redshift  $z$  and  $\omega_{\Lambda non} < -1$  indicates a decreasing negative behavior. We also notice a decreasing behavior in Fig. b. The EoS for both cases of DE does not cross the phantom barrier ( $\omega_{\Lambda} < -1$ ) and the model does not exhibit a quintessence-like behavior for various values of  $n = 5$  (black-solid),  $n = 5.2$  (blue-dashed) and  $n = 5.4$  (red-dashed), over the considered range of redshift.

The pressure ( $P$ ) of DE could be defined by using the scalar field and the interactions between DE and DM as [Pasqua, A et al. (2014)]:

$$p = \frac{\gamma}{2}(\dot{\phi})^2 - V(\phi(t)) + 16Hf'(\phi(t))\ddot{\phi} + 8H^2 \left( f'(\phi(t))\dot{\phi} + f''(\phi(t))(\dot{\phi})^2 \right), \quad (5.18)$$

where the scalar field is given by  $\phi(t) = a(t)^n \phi_o$  and  $\phi_o$  is a positive constant. The prime indicates a derivative with respect to  $\phi$  Pasqua, A et al. (2014). Here  $f'(\phi) = \frac{df}{d\phi}$  and  $f''(\phi) = \frac{d^2f}{d\phi^2}$  [Pasqua, A et al. (2014); Jawad, A et al. (2017)].

The total energy density of DE and DM  $\rho + \rho_m$  is [Pasqua, A et al. (2014)]:

$$\rho + \rho_m = \frac{\gamma}{2}(\dot{\phi})^2 + V(\phi) - 24H^3 f'(\phi)\dot{\phi}. \quad (5.19)$$

The potential of the interacting DE is given by solving Eq.(5.19) [Pasqua, A et al. (2014)]:

$$V(\phi(t)) = \rho_m + \rho - \frac{\gamma}{2}(\dot{\phi})^2 + 24f'(\phi(t))\dot{\phi}H^3. \quad (5.20)$$

To reconstruct the potential of the interacting DE as a function of the cosmic time  $t$ , we solve Eq.(2.8) with Eq.(5.7) in Eq.(5.20), then [Pasqua, A et al. (2014)]:

$$V(\phi(t)) = \rho_{mo}a^{-3(1+\delta)} + 3c^2 \left( \dot{H} + 2H^2 \right) - \frac{\gamma}{2}(\dot{\phi})^2 + 24f'(\phi(t))\dot{\phi}H^3. \quad (5.21)$$

## 6 Squared Speed Of The Sound

Let us study an important parameter in cosmology which is used to check the stability of any DE model. This quantity is known as ‘squared speed of sound’  $v_s^2$ .

The general definition is assigned as [Pasqua, A et al. (2016); Zadeh, MA and Sheykhi, A. (2018)]:

$$v_s^2 = \frac{p_{tot}}{\rho_{tot}} = \frac{\dot{p}}{\dot{\rho} + \dot{\rho}_m}, \quad (6.1)$$

where  $p_{tot} = p$  is the total pressure and  $\rho_{tot} = \rho_m + \rho_{mo}$  is the total energy density of the DE model. The sign of  $v_s^2$  plays an important role in determining the stabilization of DE model. If  $v_s^2 < 0$ , it means that the model is unstable. Moreover, if  $v_s^2 > 0$ , the model is stable [Zadeh, MA and Sheykhi, A. (2018)].

**Intermediate inflation:**

The  $v_s^2$  behavior is presented in Fig. 7:

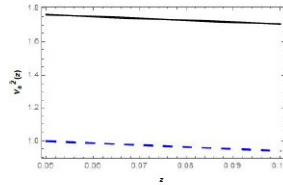


Figure 7: The behavior of  $v_s^2(z)$  as a function of red-shift  $z$  for the intermediate inflation for  $n = 5$  (black-solid),  $n = 5.2$  (blue-dashed) and  $n = 5.4$  (red-dashed).

The behavior of squared speed of sound parameter for different values of  $n$  is shown in Fig. 7. We find a decreased function remains positive when  $n = 5$  (black-solid) and  $n = 5.2$  (blue-dashed) at the considered range from  $z = 0.05$  to  $z = 0.1$ . This result indicates the stability of the model when  $v_s^2 > 0$ .

**Logamediate inflation:**

The behavior of  $v_s^2$  is revealed in Fig. 8:

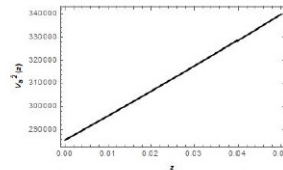


Figure 8: The behavior of  $v_s^2(z)$  as a function of red-shift  $z$  for the logamediate inflation, for  $n = 5$  (black-solid).

In Fig. 8, the behavior of  $v_s^2(z)$  as a function of redshift  $z$  in case of logamediate expansion. It represents an increasing behavior at positive level for  $n = 5$  (black-solid) and this is from  $z = 0.001$  to  $z = 0.05$ . The stability behavior of universe happens when  $v_s^2 > 0$ , independently on the values of  $n$ .

**Emergent inflation:**

The behavior of  $v_s^2$  is expressed in Fig. 9:

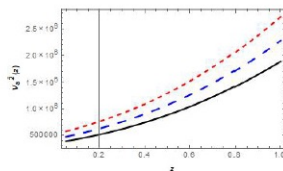


Figure 9: The behavior of  $v_s^2(z)$  as a function of red-shift  $z$  for the emergent inflation for  $n = 5$  (black-solid),  $n = 5.2$  (blue-dashed) and  $n = 5.4$  (red-dashed).

Fig. 9 clarifies an increasing positive behavior for all values of  $n$ , so  $v_s^2 > 0$  represents the stability of the model.

## 7 Generalized Second Law of Thermodynamics

Bekenstein and Hawking determined that the entropy of black hole is proportional to its event horizon [Bekenstein, JD. (1973)] which leads to GSLT for black hole physics. This law can be defined as the entropy of black hole and its area is always increasing. According to GSLT, the entropy of horizon and entropy of matter sources inside horizon does not decrease with respect to time.

$$\frac{dS_{\text{horizon}}}{dt} + \frac{dS_{\text{inside}}}{dt} \geq 0, \quad (7.1)$$

$$\dot{S}_{\text{tot}} = \dot{S}_h + \dot{S}_{\text{in}} \geq 0, \quad (7.2)$$

where  $\dot{S}_h$  is the entropy of horizon and  $\dot{S}_{\text{in}}$  is the entropy of matter inside horizon.

### 7.1 Entropy of matter inside horizon

From Gibb's equation, we have [Iqbal, A and Jawad, A. (2018)]

$$T_{\text{emp}} dS_{\text{in}} = P_{\text{eff}} dV_{\text{ol}} + dE_{\text{in}}. \quad (7.3)$$

The IR cut-off of the universe  $R_h$  is equal to the apparent horizon [Iqbal, A and Jawad, A. (2018)]:

$$R_h = \frac{1}{H}, \quad (7.4)$$

and the temperature inside the horizon is evidenced as follow [Iqbal, A and Jawad, A. (2018)]:

$$T_{\text{emp}} = \frac{1}{2\pi R_h}. \quad (7.5)$$

The volume of the system is [Iqbal, A and Jawad, A. (2018)]:

$$V_{\text{ol}} = \frac{4\pi}{3} R_h^3. \quad (7.6)$$

The internal energy of system is [Iqbal, A and Jawad, A. (2018)]:

$$E_{\text{in}} = \rho_{\text{eff}} V_{\text{ol}}. \quad (7.7)$$

Now, one can put Eq.(7.4), Eq.(7.5), Eq.(7.6) and Eq.(7.7) together in Eq.(7.3) to obvious the following relation:

$$\dot{S}_{\text{in}} = 8\pi^2 R_h^3 \left( \rho_m - \frac{\dot{\rho}}{3H} \right) (\dot{R}_h - 1). \quad (7.8)$$

### 7.2 Bekenstein Entropy

Gibbons and Hawking developed the idea of Bekenstein for cosmological system, where the entropy of cosmological event horizon is proportional to horizon area. It is afforded as [Iqbal, A and Jawad, A. (2018)]:

$$S_h = \frac{A_B}{4G}. \quad (7.9)$$

The Bekenstein equation (7.9) relates qualitatively various quantities: a geometric quantity on the right hand side, the area  $A_B$  of a black hole and an information theoretic quantity on the left hand side, the entropy  $S_h$  of the black hole [Azuma, K and Subramanian, S. (2018)]. From the relation,  $A_B = 4\pi R_h^2$  and  $G = 1$ , one obtains:

$$S_h = \pi R_h^2, \quad (7.10)$$

$$\dot{S}_h = 2\pi R_h \dot{R}_h. \tag{7.11}$$

Finally, by inserting Eq.(7.8) with Eq.(7.11) in Eq.(7.2), we get [Iqbal, A and Jawad, A. (2018)]:

$$S_{tot} \dot{=} 8\pi^2 R_h^3 \left( \rho_m - \frac{\dot{\rho}}{3H} \right) (\dot{R}_h - 1) + 2\pi R_h \dot{R}_h. \tag{7.12}$$

**Intermediate inflation:**

The behavior of total entropy performs in Fig. 10 as:

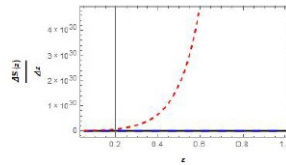


Figure 10: The behavior of  $S_{tot}(z)$  for the intermediate inflation for  $n = 1$  (black-solid),  $n = 2$  (blue-dashed) and  $n = 3$  (red-dashed).

In Fig. 10, the behavior of  $S_{tot}(z)$  as a function of redshift  $z$  in case of intermediate expansion. We have an increasing behavior at positive level ( $S_{tot}(z) > 0$ ) for  $n = 3$  (red-dashed), but for other values of  $n$ ,  $n = 1$  (black-solid) and  $n = 2$  (blue-dashed) the behavior of  $S_{tot}(z) = 0$ , from  $z = 0.05$  to  $z = 0.1$ .

**Logamediate inflation:**

The total entropy is expressed in Fig. 11 as:

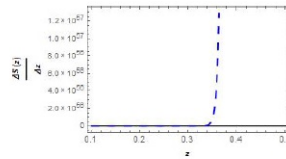


Figure 11: The behavior of  $S_{tot}(z)$  for the logamediate inflation for  $n = 5$  (black-solid),  $n = 7$  (blue-dashed) and  $n = 9$  (red-dashed).

In Fig. 11, the behavior of  $S_{tot}(z)$  as a function of redshift  $z$  for the logamediate expansion. We notice an increasing positive behavior for  $n = 7$  (blue-dashed), but for  $n = 5$  (black-solid), the behavior of  $S_{tot}(z) = 0$ , over the considered range of redshift.

**Emergent inflation:**

The behavior of total entropy is realized in Fig. 12 as:

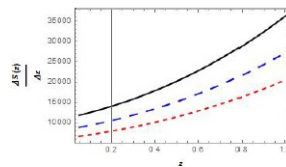


Figure 12: The behavior of  $S_{tot}(z)$  for the emergent inflation for  $n = 5$  (black-solid),  $n = 5.2$  (blue-dashed) and  $n = 5.4$  (red-dashed).

Fig. 12 presents an increasing behavior at positive level, where  $S_{tot}(z) > 0$  for all values of  $n$  from  $z = 0.05$  to  $z = 1$ .

For three cases, we have an increasing positive behavior.  $S_{tot}(z) > 0$  indicates the GSLT is verified.

## 8 CONCLUSIONS

In this paper, we have studied a new modified gravity theory called  $f(T)$  model together with different choices of scale factors to explain the cosmological accelerated expansion behavior of the universe with physical significance. During our study in each model, we used these values of physical parameters we have. For the intermediate inflation,  $u = -2$ ,  $B = 0.02$ ,  $\phi_o = 0.14$ ,  $\rho_{mo} = 0.23$ ,  $\delta = 0.05$ ,  $z_o = 1$  and  $H_o = 74$ . For the logamediate inflation,  $A = 10^{-3}$ ,  $\phi_o = 0.14$ ,  $\rho_{mo} = 0.23$  and  $\delta = 0.05$ . For the emergent inflation,  $a_o = 0.12$ ,  $\mu = 0.3$ ,  $\lambda = 0.3$ ,  $\phi_o = 0.14$ ,  $\rho_{mo} = 0.23$  and  $\delta = 0.05$ .

- a In the section (4), the characteristics of the universe represented by the deceleration parameter, the jerk parameter, the kerk parameter and the lerk parameter. The negative value of  $q$  describes the accelerated expansion of the universe, while the value of jerk parameter is helpful for distinguishing the different accelerating models, also  $j$  can indicate the dynamics of the universe acceleration. The snap parameter ( $\kappa$ ) depends on the fourth time derivatives of the scale factor and is known as kerk parameter. The lerk parameter  $l$  depends on the fifth time derivative of scale factor.
- b In the section (5), We have studied the behavior of the EoS parameter of DE for both non-interacting and interacting DE sectors. We notice that the values of the  $\Lambda$ CDM model can be assumed to be greater, equals or less than  $-1$ . Assuming  $-1 \leq \omega_\Lambda \leq 1$  indicates quintessence-like behavior, and for  $\omega_\Lambda \leq -1$  represents phantom-like behavior. In Figs. ??, 5 and ?? the EoS parameter has a decreasing behavior for both  $\omega_{\Lambda non}$  and  $\omega_{\Lambda inter}$  at negative part that indicates a phantom-like behavior.
- c In the section (6), the squared speed of the sound determines if the model is stable or not. We notice that the model is either stable or unstable depending on the values of the parameters. In Figs. ??, ?? and ??, the  $v_s^2 > 0$  indicates a stable DE model.
- d In the section (7), the GSLT proves the validity of thermal entropy and the increasing behavior of the black-hole's entropy refers to  $S_{tot}(z) > 0$ .

. These results are in agreement with the data of [Pasqua, A et al. (2014); Chattopadhyay, S and et al. (2014); Pasqua, A et al. (2016)], which proved that our model in the framework of  $f(T)$  modified gravity leads to an accelerated universe.

## References

- Arabsalmani, Maryam and Sahni, Varun. (2011). Large-scale Structure in  $f(T)$  Gravity. Physical Review D. 83(4), 043501.
- Azuma, Koji and Subramanian, Sathyawageeswar. (2018). Do black holes store negative entropy?. arXiv preprint arXiv:1807.06753.
- Barrow, John D and Nunes, NJ. (2007). Dynamics of "logamediate" inflation. Physical Review D. 76(4), 043501.
- Bekenstein, Jacob D. (1973). Black holes and entropy. Physical Review D. 7(8), 2333.
- Bilic, Neven and Tupper, Gary B and Viollier, Raoul D. (2002). Dark matter, dark energy and the Chaplygin gas.
- Chattopadhyay, Surajit and Pasqua, Antonio and Aly, Ayman A. (2014). Interacting Ricci dark energy in scalar Gauss-Bonnet gravity. The European Physical Journal Plus. 129(2), 31.

- Copeland, Edmund J and Sami, Mohammad and Tsujikawa, Shinji. (2006). Dynamics of dark energy. *International Journal of Modern Physics D*. 15(11), 1753–1935.
- Daouda, M Hamani and Rodrigues, Manuel E and Houndjo, MJS. (2012). Reconstruction of  $f(T)$  gravity according to holographic dark energy. *The European Physical Journal C*. 72(2), 1893.
- Debnath, Ujjal and Chattopadhyay, Surajit and Jamil, Mubasher. (2013). Fractional action cosmology: some dark energy models in emergent, logamediate, and intermediate scenarios of the universe. *Journal of Theoretical and Applied Physics*. 7(1), 25.
- Del Popolo, Antonino and Pace, Francesco and Mota, David F. (2019). Mass-temperature relation in  $\Lambda$  CDM and modified gravity. *Physical Review D*. 100(2), 024013.
- Dkabrowski, Mariusz P. (2005). Statefinders, higher-order energy conditions, and sudden future singularities. *Physics Letters B*. 625(3-4), 184–188.
- Ellis, George FR and Maartens, Roy. (2003). The emergent universe: Inflationary cosmology with no singularity. *Classical and Quantum Gravity*. 21(1), 223.
- Gao, Changjun and Wu, Fengquan and Chen, Xuelei and Shen, You-Gen. (2009). Holographic dark energy model from Ricci scalar curvature. *Physical Review D*. 79(4), 043511.
- Ghosh R, Chattopadhyay S, Debnath U. (2012). A dark energy model with generalized uncertainty principle in the emergent, intermediate and logamediate scenarios of the universe. *International Journal of Theoretical Physics*. 51(2), 589–603.
- Iqbal, Ayesha and Jawad, Abdul. (2018). Thermodynamics of Ricci-Gauss-Bonnet Dark Energy. *Advances in High Energy Physics*. 2018.
- Jamil, Mubasher and Momeni, D and Myrzakulov, Ratbay. (2012). Attractor solutions in  $f(T)$  cosmology. *The European Physical Journal C*. 72(3), 1959.
- Jawad, Abdul and Rani, Shamaila and Saleem, M. (2017). Cosmological study of reconstructed  $f(T)$  models. *Astrophysics and Space Science*. 362(4), 63.
- Karami, K and Khaledian, MS and Felegary, F and Azarmi, Z. (2010). Interacting new agegraphic tachyon, K-essence and dilaton scalar field models of dark energy in non-flat universe. *Physics Letters B*. 686(4-5), 216–220.
- Karami, K and Abdolmaleki, A. (2012). Generalized second law of thermodynamics in  $f(T)$  gravity. *Journal of Cosmology and Astroparticle Physics*. 2012(04), 007.
- Li, Baojiu and Sotiriou, Thomas P and Barrow, John D. (2011). Large-scale Structure in  $f(T)$  Gravity. *Physical Review D*. 83(10), 104017.
- Macdonald, Alan. (2006). Comment on "The Cosmic Time in Terms of the Redshift", by Carmeli et al. arXiv preprint gr-qc/0606038.
- Nojiri, Shin'ichi and Odintsov, Sergei D. (2007). Introduction to modified gravity and gravitational alternative for dark energy. *International Journal of Geometric Methods in Modern Physics*. 4(01), 115–145.
- Pasqua, Antonio and Chattopadhyay, Surajit and Khurshudyan, Martiros and Aly, Ayman A. (2014). Behavior of Holographic Ricci Dark Energy in Scalar Gauss-Bonnet Gravity for Different Choices of the Scale Factor. *International Journal of Theoretical Physics*. 53(9), 2988–3013.



- Pasqua, Antonio and Assaf, Khudhair A and Aly, Ayman A. (2014). Power law and logarithmic entropy corrected Ricci dark energy models in Brans-Dicke chameleon cosmology. *International Journal of Theoretical Physics*. 53(2), 566–578.
- Pasqua, Antonio and Chattopadhyay, Surajit and Beesham, Aroonkumar. (2016). A look into the cosmological consequences of a dark energy model with higher derivatives of  $H$  in the framework of Chameleon Brans-Dicke Cosmology. arXiv preprint arXiv:1607.03384.
- Tajahmad, Behzad. (2018). Reconstruction of  $F(T)$  gravity in homogeneous backgrounds. arXiv preprint arXiv:1812.10339.
- Xu, Lixin and Lu, Jianbo and Li, Wenbo. (2009). Generalized holographic and Ricci dark energy models. *The European Physical Journal C*. 64(1), 89.
- Yousaf, Z and Bamba, Kazuharu and Bhatti, MZ and Ghafoor, U. (2019). Charged gravastars in modified gravity. *Physical Review D*. 100(2), 024062.
- Zadeh, M Abdollahi and Sheykhi, A. (2018). Stability of HDE model with sign-changeable interaction in Brans-Dicke theory. arXiv preprint arXiv:1804.10843.

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