

**A Weibull-Gompertz Makeham Distribution with Properties and Application to Cancer**

**Data**

**Abstract**

The article presents an extension of the Gompertz Makeham distribution using the Weibull-G family of continuous probability distributions proposed by Tahir *et al.* (2016a). This new extension generates a more flexible model called Weibull-Gompertz Makeham distribution. Some statistical properties of the distribution which include the moments, survival function, hazard function and distribution of order statistics were derived and discussed. The parameters were estimated by the method of maximum likelihood and the distribution was applied to a bladder cancer data. Weibull-Gompertz Makeham distribution performed best (AIC = -6.8677, CAIC = -6.3759, BIC = 7.3924) when compared with other existing distributions of the same family to model bladder cancer data.

**Keywords:** Gompertz-Makeham Distribution, Weibull-Gompertz Makeham Distribution, hazard function, survival function, cancer.

**1 Introduction**

The Gompertz-Makeham distribution (GMD) was introduced by Makeham in 1860 (Makeham (1860)). It is an extended model of the Gompertz probability distribution that was introduced by Gompertz in 1825 (Gompertz (1825)). The GMD is a continuous probability distribution that has been widely used in survival analysis, modelling human mortality, constructing actuarial tables and growth models. It has been recently used in many fields of sciences including actuaries, biology, demography, gerontology, and computer science.

A comprehensive review of the history and theory of the GMD can be found in Marshall and Olkin (2007). Golubev (2004) emphasizes the practical importance of this probability distribution. Detailed information about the GM distribution, its mathematical and statistical properties, and its applications can be found in Johnson et al. (1995) and Dey et al. (2018).

The probability density function (pdf) and cumulative distribution function (cdf) of the Gompertz-Makeham distribution are defined as:

$$G(x) = 1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \tag{1}$$

and

36 
$$g(x) = \left[ \theta + \alpha e^{\beta x} \right] e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \quad (2)$$

37 respectively.

38 For  $x > 0, \alpha, \beta, \theta > 0$ , where  $\theta$  is the scale parameter and  $\alpha$  and  $\beta$  are the shape parameters of  
 39 Gompertz-Makeham distribution.

40 There are families of distributions proposed by different researchers that are used in extending  
 41 other distributions to produce compound distributions with better performance. These families  
 42 among others include the beta generalized family (Beta-G) by Eugene et al. (2002), the  
 43 Kumaraswamy-G by Cordeiro and de Castro (2011), Transmuted family of distributions by  
 44 Shaw and Buckley (2007), Exponentiated T-X by Alzaghal et al. (2013), Exponentiated-G (EG)  
 45 by Cordeiro et al. (2013), Logistic-G by Torabi and Montazari (2014), Logistic-X by Tahir et al.  
 46 (2016b), Weibull-X by Alzaatreh et al. (2013), Weibull-G by Bourguignon et al. (2014), a new  
 47 Weibull-G family by Tahir et al. (2016a), a Lomax-G family by Cordeiro et al. (2014), a new  
 48 generalized Weibull-G family by Cordeiro et al. (2015) and Beta Marshall-Olkin family of  
 49 distributions by Alizadeh et al. (2015) etc.

50 Recently, many authors have extended the Gompertz-Makeham distribution. Chukwu and  
 51 Ogunde (2016) introduced and studied the Kumaraswamy Gompertz Makeham distribution. El-  
 52 Bar (2017) used the quadratic rank transmutation map by Shaw and Buckley (2007) to defined  
 53 and study the transmuted Gompertz Makeham distribution with useful discussions as well as  
 54 applications.

55 Hence, the aim of this article is to introduce another extension of the Gompertz Makeham model,  
 56 a new continuous distribution called Weibull-Gompertz Makeham distribution (WGMD) from  
 57 the proposed family by Tahir et al. (2016a). The rest of this article is arranged as follows: the  
 58 definition of the new distribution will be presented with its plots and some properties. These are  
 59 followed by the reliability functions, the order statistics for the distribution and the maximum  
 60 likelihood estimates (MLEs) of the unknown parameters. The last part involves the application of  
 61 the proposed model with other models to a lifetime dataset and the conclusion.  
 62

63

## 64 **2 Materials and Methods**

### 65 **2.1 Construction of Weibull-Gompertz Makeham Distribution (WGMD)**

66 This section defines the cdf and pdf of the Weibull-Gompertz Makeham distribution (WGMD)  
 67 using the family of distributions proposed by Tahir et al. (2016a), which has been used by other  
 68 authors including Ieren et al. (2018). According to Tahir et al. (2016a), the function for defining  
 69 the cdf and pdf of any Weibull-based continuous distribution is given as:

$$F(x) = \int_0^{-\log[G(x)]} abt^{b-1} e^{-at} dt = e^{-a\{-\log[G(x)]\}^b} \quad (3)$$

71 and

$$f(x) = ab \frac{g(x)}{G(x)} \{-\log[G(x)]\}^{b-1} e^{-a\{-\log[G(x)]\}^b} \quad (4)$$

73 respectively, where  $g(x)$  and  $G(x)$  are the pdf and cdf of any continuous distribution to be  
 74 generalized respectively. The parameters,  $\alpha$  and  $\beta$  are the two additional new parameters  
 75 responsible for the scale and shape of the distribution respectively.

76 Using equation (1) and (2) in (3) and (4) and simplifying, the cdf and pdf of the WGMD of a  
 77 random variable X can be obtained as:

$$F(x) = e^{-a\left\{-\log\left[1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)}\right]\right\}^b} \quad (5)$$

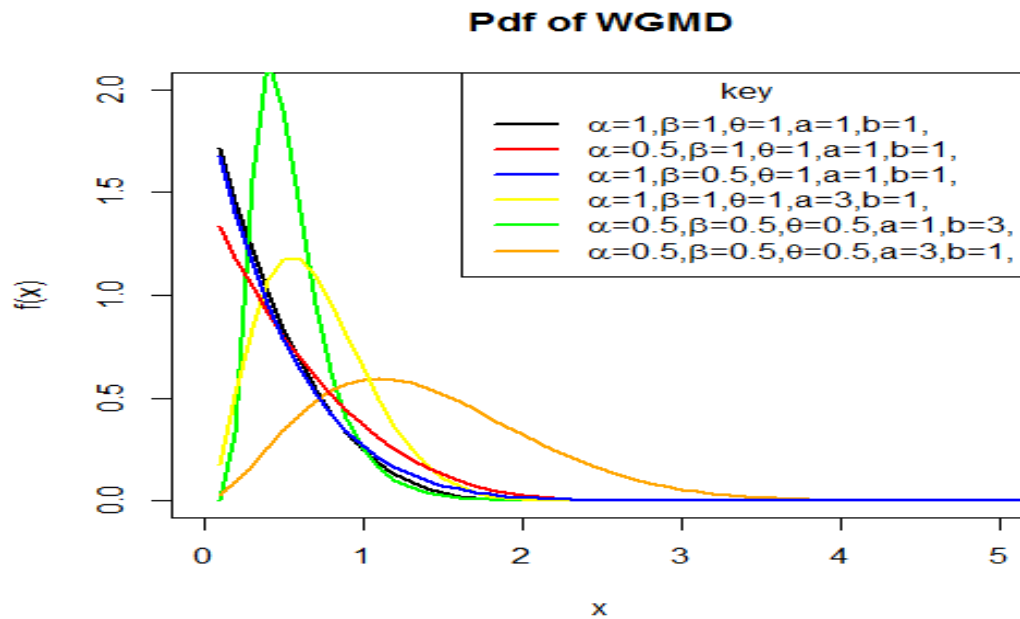
79 and

$$f(x) = \frac{ab\left(\theta + \alpha e^{\beta x}\right) e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \left(-\log\left[1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)}\right]\right)^{b-1}}{\left[1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)}\right] e^{a\left(-\log\left[1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)}\right]\right)^b}} \quad (6)$$

81 respectively.

82 For  $x > 0; a, b, \theta, \alpha, \beta > 0$ ; where  $a, b, \theta, \alpha$  and  $\beta$  are the parameters of the WGMD.

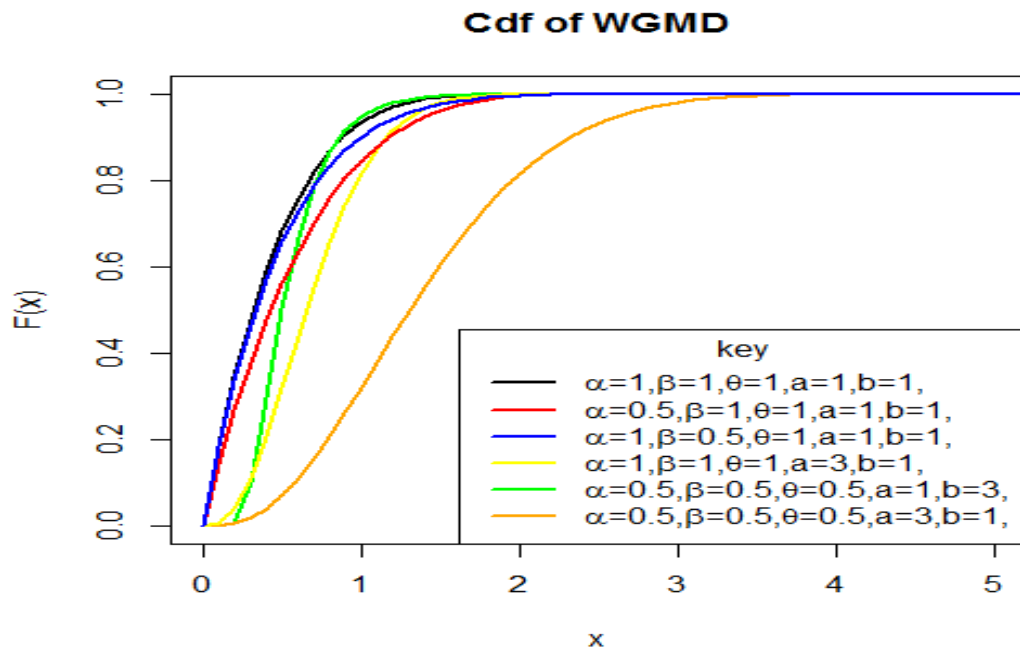
83 The following is a graphical representation of the pdf and cdf of the WGMD using arbitrary  
 84 values of the parameters  $a, b, \theta, \alpha$  and  $\beta$ .



85

86 **Figure 1:** A plot of PDF of the WGMD for varying parameter values

87 It is observed in Figure 1 that the WGMD is a positively skewed distribution and can take  
 88 various forms. This means that distribution can be very useful for datasets that are skewed.



89

90 **Figure 2:** A plot of CDF of the WGMD for varying parameter values

91 From the above cdf plot, the cdf increases when X increases, and approaches 1 when X becomes  
 92 large or tends to infinity as expected.

93  
 94 **3 Properties**

95 In this section, we defined and discuss some properties of the WGMD distribution.

96 **3.1 Moments**

97 Let X denote a continuous random variable, the n<sup>th</sup> moment of X is given by;

98 
$$\mu'_n = E[X^n] = \int_0^{\infty} x^n f(x) dx \quad (7)$$

99 Considering f(x) to be the pdf of the WGMD as given in equation (6)

100 
$$\mu'_n = E[X^n] = \int_0^{\infty} x^n f(x) dx$$

101 Recall that from equation (6),

102 
$$f(x) = \frac{ab \left( \theta + \alpha e^{\beta x} \right) e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^{b-1}}{\left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] e^{a \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^b}} \quad (8)$$

103  
 104 To simplify the pdf in (8) above, we carryout the following operations:

Let

105 
$$A = e^{-a \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^b}$$

Then, using a power series expansion for A, we can write A as:

106  
 107 
$$A = e^{-a \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^b} = \sum_{i=0}^{\infty} \frac{(-1)^i a^i}{i!} \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^{bi}$$

108 Substituting for the expansion above in equation (8), we have;

109 
$$f(x) = \frac{ab \left( \theta + \alpha e^{\beta x} \right) e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \sum_{i=0}^{\infty} \frac{(-1)^i a^i}{i!} \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^{bi}}{\left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^{-(b-1)}}$$

$$f(x) = \frac{ab \left( \theta + \alpha e^{\beta x} \right) e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \sum_{i=0}^{\infty} \frac{(-1)^i a^i}{i!} \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^{bi+(b-1)}}{\left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]} \quad (9)$$

$$B = \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^{bi+(b-1)}$$

Also, let

Now, considering the following formula from Tahir et al. (2016b) and Ieren et al. (2018) which

holds for B for  $i \geq 1$ , then B can be written as follows:

$$\left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^{bi+(b-1)} = \sum_{k,l=0}^{\infty} \sum_{j=0}^k \frac{(-1)^{j+k+l} (b(i+1))}{(b(i+1)-1-j)} \binom{k-(b(i+1)-1)}{k} \binom{k}{j} \binom{(b(i-1)+1)+k}{l} P_{j,k} \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^l \quad (10)$$

Where for (for  $j \geq 0$ )  $P_{j,0} = 1$  and (for  $k = 1, 2, \dots$ )

$$P_{j,k} = k^{-1} \sum_{m=1}^k (-1)^m \frac{[m(j+1)-k]}{(m+1)} P_{j,k-m} \quad (11)$$

Combining equation (10) and (11) and inserting the above power series in equation (9) and simplifying, it gives:

$$f(x) = ab \sum_{i=0}^{\infty} \frac{(-1)^i a^i}{i!} \sum_{k,l=0}^{\infty} \sum_{j=0}^k \frac{(-1)^{j+k+l} (b(i+1))}{(b(i+1)-1-j)} \binom{k-(b(i+1)-1)}{k} \binom{k}{j} \binom{(b(i+1)-1)+k}{l} P_{j,k} \left( \theta + \alpha e^{\beta x} \right) e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^{l-1}$$

$$f(x) = b \sum_{i=0}^{\infty} \sum_{k,l=0}^{\infty} \sum_{j=0}^k \frac{(-1)^{i+j+k+l} a^{i+1} (b(i+1))}{i! (b(i+1)-1-j)} \binom{k-(b(i+1)-1)}{k} \binom{k}{j} \binom{(b(i+1)-1)+k}{l} P_{j,k} \left( \theta + \alpha e^{\beta x} \right) e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^{l-1} \quad (12)$$

Now, if  $l$  is a positive non-integer, we can expand the last term in (12) as:

$$\left[ 1 - e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^{l-1} = \sum_{m=0}^{\infty} (-1)^m \binom{l-1}{m} \left[ e^{-\theta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^m \quad (13)$$

Therefore,  $f(x)$  becomes:

$$f(x) = b \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{k,l=0}^{\infty} \sum_{j=0}^k \frac{(-1)^{i+j+k+l+m} a^{i+1} (b(i+1))}{i! (b(i+1)-1-j)} \binom{l-1}{m} \binom{k-(b(i+1)-1)}{k} \binom{k}{j} \binom{(b(i+1)-1)+k}{l} P_{j,k} \left( \theta + \alpha e^{\beta x} \right) e^{-(m+1)\left(\theta x + \frac{\alpha}{\beta} (e^{\beta x} - 1)\right)}$$

$$f(x) = b \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{k,l=0}^{\infty} \sum_{j=0}^k \frac{(-1)^{i+j+k+l+m} a^{i+1} (b(i+1)) (l-1) \binom{l-1}{m} \binom{k-(b(i+1)-1)}{k} \binom{k}{j} \binom{(b(i+1)-1)+k}{l}}{e^{-\frac{\alpha}{\beta}(m+1)} i! (b(i+1)-1-j)} P_{j,k} \left( \theta + \alpha e^{\beta x} \right) e^{-\theta(m+1)x} e^{-\frac{\alpha}{\beta}(m+1)e^{\beta x}}$$

Using power series expansion on the last term in equation (14), we have

$$e^{-\frac{\alpha}{\beta}(m+1)e^{\beta x}} = \sum_{r=0}^{\infty} \frac{(-1)^r \alpha^r (m+1)^r}{r! \beta^r} e^{r\beta x}$$

Now, substituting equation (15), the power series expansion in equation (14) above, one gets:

$$f(x) = b \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{k,l=0}^{\infty} \sum_{j=0}^k \frac{(-1)^{i+j+k+l+m+r} \alpha^r a^{i+1} (m+1)^r (b(i+1)) (l-1) \binom{l-1}{m} \binom{k-(b(i+1)-1)}{k} \binom{k}{j} \binom{(b(i+1)-1)+k}{l}}{e^{-\frac{\alpha}{\beta}(m+1)} r! \beta^r i! (b(i+1)-1-j)} P_{j,k} \left( \theta + \alpha e^{\beta x} \right) e^{-[\theta(m+1)-r\beta]x}$$

Now, let

$$\eta_{i,j,k,l,m,r} = b \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{k,l=0}^{\infty} \sum_{j=0}^k \frac{(-1)^{i+j+k+l+m+r} \alpha^r a^{i+1} (m+1)^r (b(i+1)) (l-1) \binom{l-1}{m} \binom{k-(b(i+1)-1)}{k} \binom{k}{j} \binom{(b(i+1)-1)+k}{l}}{e^{-\frac{\alpha}{\beta}(m+1)} r! \beta^r i! (b(i+1)-1-j)} P_{j,k}$$

This implies that:

$$f(x) = \eta_{i,j,k,l,m,r} \left\{ \theta e^{-[\theta(m+1)-r\beta]x} + \alpha e^{-[\theta(m+1)-\beta(1+r)]x} \right\}$$

Hence,

$$\mu_n = E(X^n) = \int_0^{\infty} x^n f(x) dx = \int_0^{\infty} x^n \left( \eta_{i,j,k,l,m,r} \left\{ \theta e^{-[\theta(m+1)-r\beta]x} + \alpha e^{-[\theta(m+1)-\beta(1+r)]x} \right\} \right) dx$$

$$\mu_n = \int_0^{\infty} x^n f(x) dx = \eta_{i,j,k,l,m,r} \left[ \theta \int_0^{\infty} x^n e^{-[\theta(m+1)-r\beta]x} dx + \alpha \int_0^{\infty} x^n e^{-[\theta(m+1)-\beta(1+r)]x} dx \right]$$

Also, using integration by substitution method in equation (17) gives the following:

$$\text{Let } u_1 = [\theta(m+1) - r\beta]x \Rightarrow x = \frac{u_1}{\theta(m+1) - r\beta}; \frac{du_1}{dx} = \theta(m+1) - r\beta \text{ and } dx = \frac{du_1}{\theta(m+1) - r\beta}$$

141 Let  $u_2 = [\theta(m+1) - \beta(1+r)]x \Rightarrow x = \frac{u_2}{\theta(m+1) - \beta(1+r)}$ ;  $\frac{du_2}{dx} = \theta(m+1) - \beta(1+r)$  and

142 
$$dx = \frac{du_2}{\theta(m+1) - \beta(1+r)}$$

143 Substituting for  $u$ ,  $x$  and  $dx$  in equation (17) and simplifying gives:

144 
$$\mu'_n = \eta_{i,j,k,l,m,r} \left[ \frac{\theta}{(\theta(m+1) - r\beta)^{n+1}} \int_0^\infty u_1^{n+1-1} e^{-u_1} du_1 + \frac{\alpha}{(\theta(m+1) - \beta(1+r))^{n+1}} \int_0^\infty u_2^{n+1-1} e^{-u_2} du_2 \right] \quad (18)$$

145 Again recall that  $\int_0^\infty t^{n-1} e^{-t} dt = \Gamma(n)$  and that  $\int_0^\infty t^n e^{-t} dt = \int_0^\infty t^{n+1-1} e^{-t} dt = \Gamma(n+1)$

146 Thus, the  $n^{\text{th}}$  ordinary moment of X for the WGMD is given as follows:

147 
$$\mu'_n = \eta_{i,j,k,l,m,r} \left[ \frac{\theta \Gamma(n+1)}{(\theta(m+1) - r\beta)^{n+1}} + \frac{\alpha \Gamma(n+1)}{(\theta(m+1) - \beta(1+r))^{n+1}} \right] \quad (19)$$

149  
150 **3.2 The Mean**

151 The mean of the WGMD can be obtained from the  $n^{\text{th}}$  moment of the distribution when  $n = 1$  as follows:

152 
$$\mu'_1 = \eta_{i,j,k,l,m,r} \left[ \frac{\theta}{(\theta(m+1) - r\beta)^2} + \frac{\alpha}{(\theta(m+1) - \beta(1+r))^2} \right] \quad (20)$$

154  
155 **3.3 The Variance**

155 The  $n^{\text{th}}$  central moment or moment about the mean of X, say  $\mu_n$ , can be obtained as

156 
$$\mu_n = E[X - \mu'_1]^n = \sum_{i=0}^n (-1)^i \binom{n}{i} \mu_1^i \mu'_{n-i} \quad (21)$$

157 The variance of X for WGMD is obtained from the  $n^{\text{th}}$  central moment when  $n = 2$ , that is, the  
158 variance of X is the  $n^{\text{th}}$  central moment of order two ( $n = 2$ ) and is given as follows:



159 
$$Var(X) = E[X^2] - \{E[X]\}^2 \quad (22)$$

160 
$$Var(X) = \mu_2' - \{\mu_1'\}^2$$

161 
$$Var(X) = \eta_{i,j,k,l,m,r} \left[ \frac{2\theta}{(\theta(m+1)-r\beta)^3} + \frac{2\alpha}{(\theta(m+1)-\beta(1+r))^3} \right] - \left\{ \eta_{i,j,k,l,m,r} \left[ \frac{\theta}{(\theta(m+1)-r\beta)^2} + \frac{\alpha}{(\theta(m+1)-\beta(1+r))^2} \right] \right\}^2 \quad (23)$$

162 The coefficients variation, skewness and kurtosis measures can also be calculated from the non-  
163 central moments using some well-known relationships.

164

### 165 3.4 Moment Generating Function

166 The mgf of a random variable X can be obtained by

167 
$$M_x(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} f(x) dx \quad (24)$$

168 Using power series expansion in equation (24) and simplifying the integral gives;

169 
$$M_x(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mu_n' = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left\{ \eta_{i,j,k,l,m,r} \left[ \frac{\theta \Gamma(n+1)}{(\theta(m+1)-r\beta)^{n+1}} + \frac{\alpha \Gamma(n+1)}{(\theta(m+1)-\beta(1+r))^{n+1}} \right] \right\} \quad (25)$$

170 where n and t are constants, t is a real number and  $\mu_n'$  denotes the n<sup>th</sup> ordinary moment of X .

171

### 172 3.5 Characteristics Function

173 The characteristics function of a random variable X is given by;

174 
$$\phi_x(t) = E[e^{itx}] = E[\cos(tx) + i \sin(tx)] = E[\cos(tx)] + E[i \sin(tx)] \quad (26)$$

175

176 Simple algebra and power series expansion proves that

177 
$$\phi_x(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mu_{2n}' + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \mu_{2n+1}' \quad (27)$$

178 Where  $\mu_{2n}'$  and  $\mu_{2n+1}'$  are the moments of X for n=2n and n=2n+1 respectively and can be  
179 obtained from  $\mu_n'$  in equation (20).

180

### 181 4 Some Reliability Functions

182 In this section, the survival and hazard functions from the WGMD are presented with adequate  
183 plots and their discussions.

184

#### 185 4.1 The Survival Function

186 The survival function as the name implies describes the probability that a component or an  
187 individual will not fail after a given time. It is mathematically given as:

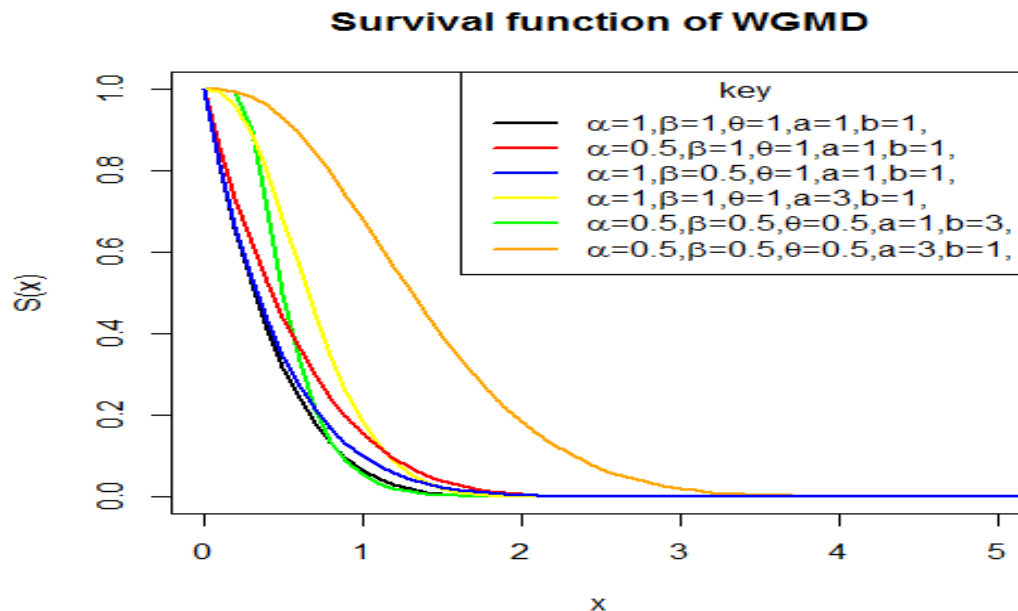
188 
$$S(x) = 1 - F(x) \tag{28}$$

189 Taking  $F(x)$  to be the cdf of the WGMD, substituting and simplifying (28) above, we get the  
190 survival function for the WGMD as:

191 
$$S(x) = 1 - e^{-a \left( -\log \left[ 1 - e^{-\frac{\theta x - \alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^b} \tag{29}$$

192

The following is a plot of the survival function for arbitrary parameter values.



193

194 **Figure 3:** A plot of the survival function of WGMD.

195 The figure above reveals that the probability of survival for any random variable following a  
196 WGMD which decreases as the time increases, that is, as time goes on, probability of life

197 decreases as it is expected. This shows that the WGMD would be useful for modeling most real  
 198 life situations.

199

200 **4.2 The Hazard Function**

201 Hazard function is also called failure or risk function. It describes the probability failure for a  
 202 component given an interval of time. The hazard function is defined mathematically as;

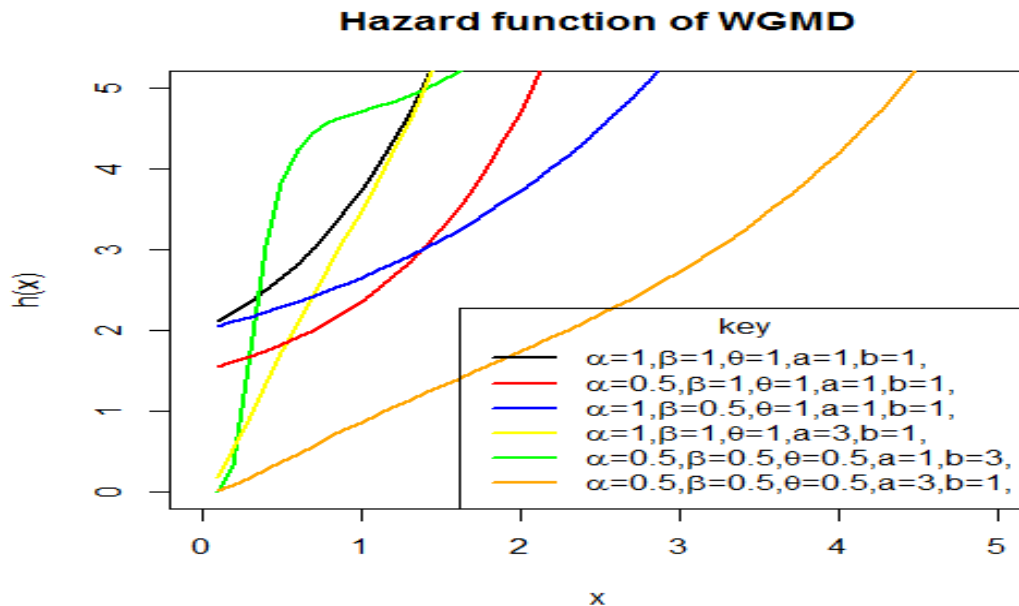
203 
$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} \tag{30}$$

204 Considering  $f(x)$  and  $F(x)$  to be the pdf and cdf of the proposed WGMD given previously, we  
 205 obtain the hazard function as:

206 
$$h(x) = \frac{ab\left(\theta + \alpha e^{\beta x}\right) e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \left(-\log\left[1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)}\right]\right)^{b-1} e^{-a\left(-\log\left[1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)}\right]\right)^b}}{\left[1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)}\right] \left[1 - e^{-a\left(-\log\left[1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)}\right]\right)^b}\right]} \tag{31}$$

207

The following is a plot of the hazard function at chosen parameter values.



208

209 **Figure 4:** A plot of the hazard function of the WGMD.

210 Figure 4 above shows the behaviour of hazard function of the WGMD. It means that the  
 211 probability of failure for any WGM random variable increases as the time or age of a subject  
 212 increases, that is, as time goes on, the probability of failure or death increases.

213

## 214 5 Order Statistics

215 Suppose  $X_1, X_2, \dots, X_n$  is a random sample from a distribution with pdf,  $f(x)$ , and  
 216  $X_{1:n} < X_{2:n} < \dots < X_{i:n}$  denote the corresponding order statistic obtained from this sample. Then  
 217 the pdf,  $f_{i:n}(x)$  of the  $i^{th}$  order statistic can be defined as;

$$218 \quad f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} f(x) F(x)^{k+i-1} \quad (32)$$

219 where  $f(x)$  and  $F(x)$  are the pdf and cdf of the Weibull Gompertz Makeham distribution  
 220 respectively.

221 Using (5) and (6), the pdf of the  $i^{th}$  order statistics  $X_{i:n}$ , can be expressed from (32) as;

$$222 \quad f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \frac{ab(\theta + \alpha e^{\beta x}) \left( -\log \left[ 1 - e^{-\frac{\theta x - \alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^{b-1}}{\left[ 1 - e^{-\frac{\theta x - \alpha}{\beta} (e^{\beta x} - 1)} \right] e^{a \left( -\log \left[ 1 - e^{-\frac{\theta x - \alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^b} e^{\frac{\theta x + \alpha}{\beta} (e^{\beta x} - 1)}} \left[ e^{-a \left( -\log \left[ 1 - e^{-\frac{\theta x - \alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^b} \right]^{i+k-1} \quad (33)$$

223 Hence, the pdf of the minimum order statistic  $X_{(1)}$  and maximum order statistic  $X_{(n)}$  of the  
 224 WGMD are given by;

$$225 \quad f_{1:n}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{ab(\theta + \alpha e^{\beta x}) \left( -\log \left[ 1 - e^{-\frac{\theta x - \alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^{b-1}}{\left[ 1 - e^{-\frac{\theta x - \alpha}{\beta} (e^{\beta x} - 1)} \right] e^{a \left( -\log \left[ 1 - e^{-\frac{\theta x - \alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^b} e^{\frac{\theta x + \alpha}{\beta} (e^{\beta x} - 1)}} \left[ e^{-a \left( -\log \left[ 1 - e^{-\frac{\theta x - \alpha}{\beta} (e^{\beta x} - 1)} \right] \right)^b} \right]^k \quad (34)$$

226 and

$$f_{n:n}(x) = n \frac{\left[ \frac{ab(\theta + \alpha e^{\beta x}) \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \right] \right)^{b-1}}{\left[ 1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \right] e^{a \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \right] \right)^b} e^{\theta x + \frac{\alpha}{\beta}(e^{\beta x} - 1)}} \right] e^{-a \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \right] \right)^b} \right]^{n-1}} \quad (35)$$

228 respectively.

229

## 230 6 Estimation of unknown Model Parameters using Maximum Likelihood Method

231 Let  $X_1, X_2, \dots, X_n$  be a sample of size 'n' independently and identically distributed random  
 232 variables from the WGMD with unknown parameters  $a, b, \alpha, \beta$  and  $\theta$  defined previously. The  
 233 pdf of the WGMD is given from (6) as:

$$234 \quad f(x) = \frac{ab(\theta + \alpha e^{\beta x}) e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \right] \right)^{b-1} e^{-a \left( -\log \left[ 1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \right] \right)^b}}{\left[ 1 - e^{-\theta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \right]}$$

235 The likelihood function is given by;

$$236 \quad L(\underline{X} | a, b, \alpha, \beta, \theta) = \frac{(ab)^n \prod_{i=1}^n \left( (\theta + \alpha e^{\beta x_i}) e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right) \prod_{i=1}^n \left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right)^{b-1} e^{-a \sum_{i=1}^n \left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right)^b}}{\prod_{i=1}^n \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right]} \quad (36)$$

237 Let the log-likelihood function,  $l = \log L(\underline{X} | a, b, \alpha, \beta, \theta)$ , therefore

$$238 \quad l = n \log a + n \log b + \sum_{i=1}^n \log \left( \theta + \alpha e^{\beta x_i} \right) - \theta \sum_{i=1}^n x_i - \frac{\alpha}{\beta} \sum_{i=1}^n (e^{\beta x_i} - 1) + (b-1) \sum_{i=1}^n \log \left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right) \\ 239 \quad - a \sum_{i=1}^n \left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right)^b - \sum_{i=1}^n \left( 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right) \quad (37)$$

240 Differentiating  $l$  partially with respect to  $a, b, \alpha, \beta$  and  $\theta$  respectively gives;

$$241 \quad \frac{\partial l}{\partial a} = \frac{n}{a} - \sum_{i=1}^n \left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right)^b \quad (38)$$

$$243 \quad \frac{\partial l}{\partial b} = \frac{n}{b} + \sum_{i=1}^n \log \left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right) - a \sum_{i=1}^n \left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right)^b \log \left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right) \quad (39)$$

$$\begin{aligned}
246 \quad \frac{\partial l}{\partial \alpha} &= \sum_{i=1}^n \left\{ \frac{e^{\beta x_i}}{\theta + \alpha e^{\beta x_i}} \right\} - \frac{1}{\beta} \sum_{i=1}^n (e^{\beta x_i} - 1) + \frac{(b-1)}{\beta} \sum_{i=1}^n \left\{ \frac{(e^{\beta x_i} - 1) e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)}}{\left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right) \left( 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right)} \right\} \\
247 \quad &- \frac{1}{\beta} \sum_{i=1}^n \left\{ \frac{(e^{\beta x_i} - 1) e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)}}{\left( 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right)} \right\} + ab \sum_{i=1}^n \left\{ \frac{\left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right)^{b-1} (e^{\beta x_i} - 1) e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)}}{\left( 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right)} \right\} \quad (40)
\end{aligned}$$

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$$\begin{aligned}
249 \quad \frac{\partial l}{\partial \beta} &= \sum_{i=1}^n \left\{ \frac{\alpha e^{\beta x_i}}{\left( \theta + \alpha e^{\beta x_i} \right)} \right\} - \frac{\alpha}{\beta} \sum_{i=1}^n \left\{ x_i e^{\beta x_i} - \beta^{-1} (e^{\beta x_i} - 1) \right\} + \frac{\alpha(b-1)}{\beta} \sum_{i=1}^n \left\{ \frac{\left( x_i e^{\beta x_i} - \beta^{-1} (e^{\beta x_i} - 1) \right) e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)}}{\left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right) \left( 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right)} \right\} \\
250 \quad &- \frac{\alpha}{\beta} \sum_{i=1}^n \left\{ \frac{\left( x_i e^{\beta x_i} - \beta^{-1} (e^{\beta x_i} - 1) \right) e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)}}{\left( 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right)} \right\} + \frac{ab\alpha}{\beta} \sum_{i=1}^n \left\{ \frac{\left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right)^{b-1} \left( x_i e^{\beta x_i} - \beta^{-1} (e^{\beta x_i} - 1) \right) e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)}}{\left( 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right)} \right\} \quad (41)
\end{aligned}$$

$$251 \quad \frac{\partial l}{\partial \theta} = \sum_{i=1}^n \left\{ \frac{1}{\left( \theta + \alpha e^{\beta x_i} \right)} \right\} - \sum_{i=1}^n x_i - (b-1) \sum_{i=1}^n \left\{ \frac{x_i e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)}}{\left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right) \left( 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right)} \right\}$$

$$252 \quad - \sum_{i=1}^n \left\{ \frac{x_i e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)}}{\left( 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right)} \right\} + ab \sum_{i=1}^n \left\{ \frac{\left( -\log \left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right] \right)^{b-1} x_i e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)}}{\left[ 1 - e^{-\theta x_i - \frac{\alpha}{\beta}(e^{\beta x_i} - 1)} \right]} \right\} \quad (42)$$

253 Equating equation (38), (39), (40), (41) and (42) to zero and solving for the solution of the non-  
254 linear system of equations gives the maximum likelihood estimates of the parameters  $a, b, \alpha, \beta$   
255 and  $\theta$  respectively. However the solution cannot be obtained analytically except numerically with  
256 the aid of suitable statistical software like Python, R, SAS, etc., when data sets are given.

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260 **7 Application to a real life dataset**

261 This section presents a dataset on the remission times (in months) of a random sample of 128  
262 bladder cancer patients with its descriptive statistics and application to some selected extensions  
263 of the Gompertz-Makeham distribution together with the classical Gompertz distribution. The  
264 performance of the Weibull Gompertz-Makeham distribution (WGMD) is compared to some  
265 families of Makeham distribution such as Kumaraswamy Gompertz Makeham distribution  
266 (KGMD), Transmuted Gompertz-Makeham distribution (TGMD), Gompertz-Makeham  
267 distribution (GMD) and the Gompertz distribution (GD).

268 The performance of the above listed models is ranked using some criteria such as the AIC  
269 (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion) and BIC  
270 (Bayesian Information Criterion). It is considered that the model with the smallest values of  
271 these statistics will be the best model to fit the data.

272 **Data set:** This data set represents the remission times (in months) of a random sample of 128  
273 bladder cancer patients. It has previously been used by Ieren and Chukwu (2018), Ieren and  
274 Kuhe (2018) and Abdullahi et al. (2018). It is summarized as follows:

275 **Table 1: Summary Statistics for the dataset**

Parameter	n	Min	$Q_1$	Median	$Q_3$	Mean	Max	Var	Skew	Kurt
Values	128	0.0800	3.348	6.395	11.840	9.366	79.05	110.425	3.3257	19.1537

n-sample size, Min-Minimum,  $Q_1$ -First Quartile,  $Q_3$ -Third Quartile, Max-Maximum, Var-Variance, Skew-Skewness, Kurt-Kurtosis

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277 From the descriptive statistics in table 1, it is observed that the data set is positively skewed with  
278 a very high coefficient of kurtosis and therefore suitable for flexible and skewed distributions.

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288 **Table 2:** The strength of the selected models using the AIC, CAIC and BIC values of the models  
 289 evaluated from the maximum likelihood estimations based on the bladder cancer data.

Distributions	Parameter estimates	AIC	CAIC	BIC	Ranks of models
<b>WGMD</b>	$\alpha = 0.006233$ $\beta = 0.005620$ $\theta = 0.006119$ $a = 0.006070$ $b = 0.004811$	-6.8677	-6.3759	7.3924	<b>1</b>
<b>KGMD</b>	$\alpha = 1.4996$ $\beta = 0.0008286$ $\theta = 3.2157$ $a = 0.1934$ $b = 0.09415$	17.3139	17.8057	31.5741	<b>3</b>
<b>TGMD</b>	$\alpha = 9.7913$ $\beta = 9.6327$ $\theta = 9.4246$ $\lambda = 0.8682$	51.5307	51.8559	62.9388	<b>5</b>
<b>GMD</b>	$\alpha = 2.04152$ $\beta = 7.8924$ $\theta = 7.9969$	29.2259	29.4194	37.7820	<b>4</b>
<b>GD</b>	$\alpha = 4.5814$ $\beta = 4.5809$	12.8378	12.9338	18.5418	<b>2</b>

290 From Table 2, comparing the values of the AIC, CAIC and BIC for each model, the WGMD has  
 291 the best performance compared to the KGMD, TGMD GMD and GD. This is due to the decision  
 292 rule which says that the distribution or model with the smallest values of the test statistics (AIC,  
 293 CAIC and BIC) is taken as the most adequate or efficient model. These values also agree with  
 294 the fact that generalizing any continuous distribution provides a compound distribution with a  
 295 better fit than the baseline distribution (Koleoso et al. (2019)).

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## 298 **8 Conclusion**

299 This article proposed a new distribution called Weibull-Gompertz Makeham distribution. The  
 300 statistical properties of the distribution have been derived and studied extensively. The model  
 301 parameters were estimated using maximum likelihood method. The distribution (WGMD) has  
 302 the best fit compared to the other four models considered in this study.

## 303 **Competing Interests**

304 Authors have declared that no competing interests exist.

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