Original Research Article

Some exact solutions of compressible and incompressible Euler equations

Abstract

In this paper, we use a surprised system to construct some exact solutions of compressible Euler equations with two and three dimension. Furthermore, we also give other exact solutions of three dimension incompressible Euler equations.

Keywords: compressible and incompressible Euler equations; cylindrical coordinate; Exact solutions.

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1 Introduction

In this paper, we are concerned with the exact solution to 2 and 3 dimension compressible Euler equations

$$\begin{cases} \partial_t \rho + div(\rho \bar{u}) = 0 \\ \partial_t(\rho \bar{u}) + div(\rho \bar{u} \otimes \bar{u}) + \nabla \bar{P}(\rho) = 0 \\ \bar{u}|_{t=0} = \bar{u}_0, \rho|_{t=0} = \rho_0 \end{cases} \tag{1.1}$$

where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, n = 2, 3$, and $\bar{u} \in \mathbb{R}^n$, ρ, $\bar{P}$ stand for the velocity, density,pressure of gases respectively. If $(\rho, \bar{u}) \in C^1$ is a solution of the above systems with $\rho \neq 0$, then it admits the following equivalent form

$$\begin{cases} \partial_t \rho + div(\rho \bar{u}) = 0 \\ \partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \bar{p}(\rho) = 0 \\ \bar{u}|_{t=0} = \bar{u}_0, \rho|_{t=0} = \rho_0 \end{cases} \tag{1.2}$$

We don't directly study (1.1) or (1.2), but demonstrate the surprising systems

$$\begin{cases} \partial_t divu + u \nabla divu = \varepsilon (divu)^2 + f(t) \\ \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ u|_{t=0} = u_0, divu|_{t=0} = divu_0 \end{cases} \tag{1.3}$$

where $f(t)$ is a function depending only on time t, and $\varepsilon = \pm 1$. Using the above equations's solutions, we can construct the solutions belonging to (1.2), and at same time, we also give some exact solutions of the 3 dimension incompressible Euler quations

$$\begin{cases} \partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{p} = 0 \\ div \tilde{u} = 0 \\ \tilde{u}|_{t=0} = \tilde{u}_0 \end{cases} \tag{1.4}$$

In fact, when $\varepsilon = -1$ and $f(t) \equiv 0$, if u, p is a solution of (1.3) with $divu \in C^1$, then by the the inverse function theorem, we have $p = q(u) = q \circ (div)^{-1}(divu) = \bar{p}(divu)$. That is,

$$\bar{u} = u, \rho = divu, \bar{p}(s) = q \circ (div)^{-1}(s) \tag{1.5}$$

is a solution of (1.2). While, $\varepsilon = 1$, (1.4) have a family solutions likely

$$\tilde{u}(t, \tilde{x}) = (u, x_{n+1} divu), \tilde{p}(t, \tilde{x}) = p(t, x) + \frac{1}{2} x_{n+1}^2 f(t) \tag{1.6}$$

So,(1.3) is a magical system.

In this paper, we considerate (1.3) with 2 and 3 dimension under cylindrical coordinate. Since 2 dimension cylindrical coordinate is contained in the 3 dimension, so we state to study (1.3) with 3 dimension. We demonstrate the axisymmetric solution

$$u(t, r, z) = u^r(t, r, z)\vec{e}_r + u^\theta(t, r, z)\vec{e}_\theta + u^z(t, r, z)\vec{e}_z \tag{1.7}$$

with

$$\vec{e}_r = (\frac{x}{r}, \frac{y}{r}, 0), \vec{e}_\theta = (-\frac{y}{r}, \frac{x}{r}, 0), \vec{e}_z = (0, 0, 1), r = \sqrt{x_1^2 + x_2^2}$$

Then,by calculating, we get the facts that

$$u \cdot \nabla = u^r \partial_r + \frac{1}{r} u^\theta \partial_\theta + u^z \partial_z,$$

and

$$divu = \frac{1}{r} u^r + u_r^r + u_z^z$$

Thus, we can reduce the axisymmetric equations

$$\begin{cases} (\frac{1}{r}u^r + u_r^r + u_z^z)_t + u^r(\frac{1}{r}u^r + u_r^r + u_z^z)_r + u^z(\frac{1}{r}u^r + u_r^r + u_z^z)_z = \varepsilon(\frac{1}{r}u^r + u_r^r + u_z^z)^2 + f(t) \\ u_t^r + u^r u_r^r - \frac{1}{r}(u^\theta)^2 + u^z u_z^r + p_r = 0 \\ u_t^\theta + u^r u_r^\theta + \frac{1}{r} u^r u^\theta + u^z u_z^\theta = 0 \\ u_t^z + u^r u_r^z + u^z u_z^z + p_z = 0 \end{cases} \tag{1.8}$$

In addition, there some works about the exact solutions of (1.2), such as [1, 2, 3, 8]. In [5], K.L. Cheun gave some blow-up solutions. In our paper, we also give some blow-up exact solutions by choosing suitable parametric functions. And Blake in [4] gave periodic structure's solutions. The same solution in our work, is given. Moreover, people also consider other solution with befitting conditions, for example [6, 7].

2 The exact solutions for $n = 2$

Under the case, (1.7) and (1.8) respectively becomes

$$u(t, r) = u^r(t, r)\vec{e}_r + u^\theta(t, r)\vec{e}_\theta \tag{2.1}$$

with $\vec{e}_r = (\frac{x}{r}, \frac{y}{r}), \vec{e}_\theta = (-\frac{y}{r}, \frac{x}{r})$, and

$$\begin{cases} (\frac{1}{r}u^r + u_r^r)_t + u^r(\frac{1}{r}u^r + u_r^r)_r = \varepsilon(\frac{1}{r}u^r + u_r^r)^2 + f(t) \\ u_t^r + u^r u_r^r - \frac{1}{r}(u^\theta)^2 + p_r = 0 \\ u_t^\theta + u^r u_r^\theta + \frac{1}{r} u^r u^\theta = 0 \end{cases} \tag{2.2}$$

Based on the above system, we know that once we have the expression of u^r, we right now use the characteristics' method to give exact expression of u^θ depending on the equation

$$u_t^\theta + u^r u_r^\theta + \frac{1}{r} u^r u^\theta = 0 \tag{2.3}$$

and also have

$$p = \int \frac{1}{r}(u^\theta)^2 dr - \int u_t^r dr - \frac{1}{2}(u^r)^2 \tag{2.4}$$

Thus, we employ the first equation in (2.2) to give some exactly type of u^r.

Write

$$\eta = \frac{1}{r}u^r + u_r^r$$

Then, by ODE theorem we have

$$u^r = \frac{e(t)}{r} + \frac{1}{r}\int r\eta dr$$

Applying the expression and the first equation in (2.2), we get

$$\eta_t + (\frac{e(t)}{r} + \frac{1}{r}\int r\eta dr)\eta_r = \varepsilon\eta^2 + f(t)$$

Let

$$\eta(t,r) = w(t,z), z = r^2,$$

then, we obtain

$$w_t + (2e(t) + \int wdz)w_z = \varepsilon w^2 + f(t).$$

Against, writing

$$\xi = 2e(t) + \int wdz,$$

then, in the end, we think about the problem

$$\xi_{zt} + \xi\xi_{zz} = \varepsilon\xi_z^2 + f(t) \tag{2.5}$$

We learn the case $\varepsilon = 1$. We build up the type solution

$$\xi(t,z) = \theta(t) + z\pi(t) + p(t)e^{zk(t)} + q(t)e^{-zk(t)}$$

After computing, we get

$$\xi(t,z) = \theta(t) + z\pi(t) + \alpha\exp\left(\int 3\pi(t) - \theta(t)k(t)dt + zk(t)\right) + \beta\exp\left(\int 3\pi(t) + \theta(t)k(t)dt - zk(t)\right)$$

and

$$f(t) = 4\alpha\beta s^2\exp\left(4\int \pi(t)dt\right) + g(t,\pi), \alpha, \beta, s \in \mathbb{R},$$

with $k(t) = s\exp(-\int \pi(t)dt)$, where $\pi(t)$ satisfies the ODE:

$$\pi' = \pi^2 + g(t,\pi), g(t,0) = 0.$$

If $\pi \equiv 0$, then $f(t) = 4\alpha\beta s^2$. Thanks to the arbitrary of α, β, s, we require that $f(t)$ is arbitrary real number. Thus, we have

$$\begin{aligned} u^r(t,r) = \frac{e(t)}{r} + \frac{r\pi(t)}{2} &+ \frac{\alpha}{2r}\exp\left(\int 3\pi(t) - \theta(t)k(t)dt + r^2k(t)\right) \\ &+ \frac{\beta}{2r}\exp\left(\int 3\pi(t) + \theta(t)k(t)dt - r^2k(t)\right) \end{aligned}$$

At the same time, we also have periodic solution $\xi(t,z)$ due to $\theta(t)$, namely that

$$\xi(t,z) = \theta(t) + kcos(\alpha z - \alpha\int\theta(t)dt + \beta) + ksin(\alpha z - \alpha\int\theta(t)dt + \beta)$$

where $f(t) = -2\alpha^2k^2, \alpha, \beta, k \in \mathbb{R}$. At once, we get other type

$$u^r(t,r) = \frac{e(t)}{r} + \frac{\sqrt{2}k}{2r}sin\left(\alpha r^2 - \alpha\int\theta(t)dt + \beta + \frac{\pi}{4}\right)$$

what is more, if $\xi(t,x)$ is a solution, then the function

$$\lambda^a\xi(\lambda^{a+b}t, \lambda^b x)$$

is also a solution with $\lambda, a, b \in \mathbb{R} - \{0\}$ and $\lambda^{2a+2b}f(t)$.

When $\varepsilon = -1$ and $f(t) \equiv 0$, (2.5) reduces

$$\begin{cases} \xi_t + \xi\xi_z = h(t) \\ \xi(0,z) = g(z) \end{cases} \tag{2.6}$$

We use characteristics of the method to build solution. Hence, we have

$$\xi(t,z) = \tilde{\xi}(t,z_0) = g(z_0) + \int_0^t h(t')dt'$$

where z_0 meets

$$z = z_0 + tg(z_0) + \int_0^t\int_0^{t'} h(t'')dt''dt'$$

Choosing suitable $g(z_0)$,so that $G(z_0) = z_0 + tg(z_0) \in C^1$, then we have

$$\xi(t,z) = g\left(G^{-1}(z - \int_0^t\int_0^{t'} h(t'')dt''dt')\right) + \int_0^t h(t')dt'$$

then by the above conversions, we get

$$u^r(t,r) = \frac{e(t)}{r} + \frac{1}{2r}g\left(G^{-1}(r^2 - \int_0^t\int_0^{t'} h(t'')dt''dt')\right).$$

No matter what ε is, we both have exact solution u^r. Next, we use the solutions to solve u^θ and p. It is obvious that $u^\theta = \frac{a}{r}$ with $a \in \mathbb{R}$ is a solution of (2.4) no matter how u^r is complex. In addition, we study the relatively difficult solution for u^θ. Let $u^r(t,r) = \frac{\phi(t)}{r} + \frac{\psi(t)}{2}r$, then using characteristics' method, we get

$$u^\theta(t,r) = \frac{g\left(\sqrt{(r^2 - 2\int_0^t\phi(t')e^{-\frac{1}{2}\int\psi(t')dt'}dt')}e^{\frac{1}{4}\int_0^t\psi(t')dt'}\right)r}{\sqrt{(r^2 - 2\int_0^t\phi(t')e^{-\frac{1}{2}\int\psi(t')dt'}dt')}e^{\frac{1}{4}\int_0^t\psi(t')dt'}}$$

where $g(\cdot) \in C^1$ and $\psi(t)$ satisfies the ODE

$$\psi' = \varepsilon\psi^2 + f(t)$$

Therefore, applying the above results, we have the following consequence.

Theorem 2.1. Let $\alpha, \beta, \gamma, \delta \neq 0$ be constants, $r = \sqrt{x^2+y^2}$, $k(t) = \gamma \exp(-\int \pi(t)dt)$, and the function $\pi(t)$ satisfies the ODE

$$\pi' = \pi^2 + g(t,\pi), g(t,0) = 0.$$

Then, three dimension incompressible Euler equations (1.4) has a class of exact solutions

$$\tilde{u}(t,x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}} u^r - \frac{\delta y}{x^2+y^2}, \frac{y}{\sqrt{x^2+y^2}} u^r + \frac{\delta x}{x^2+y^2}, -(\frac{1}{r} u^r + u_r^r) z \right) \tag{2.7}$$

$$p = -\frac{\delta^2}{2r^2} - \int u_t^r dr - \frac{1}{2}(u^r)^2 + \frac{1}{2} z^2 f(t) \tag{2.8}$$

with

$$\begin{aligned} u^r(t,r) &= \frac{e(t)}{r} + \frac{r\pi(t)}{2} + \frac{\alpha}{2r} \exp\left(\int 3\pi(t) - \theta(t)k(t)dt + r^2 k(t) \right) \\ &+ \frac{\beta}{2r} \exp\left(\int 3\pi(t) + \theta(t)k(t)dt - r^2 k(t) \right) \\ f(t) &= 4\alpha\beta\gamma^2 \exp\left(4 \int \pi(t)dt \right) + g(t,\pi) \end{aligned}$$

or

$$u^r(t,r) = \frac{e(t)}{r} + \frac{\sqrt{2}\gamma}{2r} \sin\left(\alpha r^2 - \alpha \int \theta(t)dt + \beta + \frac{\pi}{4} \right), f(t) = -2\alpha^2\gamma^2$$

In addition, it also has other kinds of exact solutions

$$\tilde{u}(t,x,y,z) = \left(\frac{x\phi(t)}{x^2+y^2} + \frac{x\psi(t)}{2} - \frac{y}{\sqrt{x^2+y^2}} u^\theta, \frac{y\phi(t)}{x^2+y^2} + \frac{y\psi(t)}{2} + \frac{x}{\sqrt{x^2+y^2}} u^\theta, -z\psi(t) \right) \tag{2.9}$$

$$p = \int \frac{1}{r}(u^\theta)^2 dr - \phi'(t) \ln r - \frac{1}{4}\psi'(t) r^2 - \frac{1}{2}(\frac{\phi(t)}{r} + \frac{\psi(t)}{2} r)^2 + \frac{1}{2} z^2 (\psi' - \psi^2) \tag{2.10}$$

with

$$u^\theta(t,r) = \frac{h\left(\sqrt{(r^2 - 2\int_0^t \phi(t') e^{-\frac{1}{2}\int \psi(t')dt'} dt') e^{\frac{1}{4}\int_0^t \psi(t')dt'}} \right) r}{\sqrt{(r^2 - 2\int_0^t \phi(t') e^{-\frac{1}{2}\int \psi(t')dt'} dt') e^{\frac{1}{4}\int_0^t \psi(t')dt'}}}.$$

Remark 2.2. About the exact solution of (1.4), there are many works,likely [9, 10, 11, 12]. In this paper, we have a great improvement than [10]. Choosing suitable parametric functions, we can get different solutions. But the energy is not finite.

Theorem 2.3. Two dimension compressible Euler equations (1.2) has a sires of exact solutions

$$\begin{aligned} \bar{u}(t,x,y) &= \left(\frac{x}{\sqrt{x^2+y^2}} u^r - \frac{\alpha y}{x^2+y^2}, \frac{y}{\sqrt{x^2+y^2}} u^r + \frac{\alpha x}{x^2+y^2} \right), \\ p(t,x,y) &= -\frac{\alpha^2}{2r^2} - \int u_t^r dr - \frac{1}{2}(u^r)^2 \end{aligned} \tag{2.11}$$

with

$$u^r(t,r) = \frac{e(t)}{r} + \frac{1}{2r} g\left(G^{-1}(r^2 - \int_0^t \int_0^{t'} h(t'')dt''dt') \right).$$

What is more, we also have other terms of solutions

$$\bar{u}(t,x,y)=\left(\frac{x\phi(t)}{x^2+y^2}+\frac{x}{2(t+\beta)}-\frac{y}{\sqrt{x^2+y^2}}u^\theta,\frac{y\phi(t)}{x^2+y^2}+\frac{y}{2(t+\beta)}+\frac{x}{\sqrt{x^2+y^2}}u^\theta\right), \quad (2.12)$$

$$p(t,x,y)=\int\frac{1}{r}(u^\theta)^2dr-\phi'(t)\ln r+\frac{r^2}{4(t+\beta)^2} \quad (2.13)$$

with

$$u^\theta(t,r)=\frac{g\left((\frac{t}{\beta}+1)^{\frac{1}{4}}\sqrt{r^2-2\int_0^t\frac{\phi(t')}{\sqrt{t+\beta}}dt'}\right)r}{(\frac{t}{\beta}+1)^{\frac{1}{4}}\sqrt{r^2-2\int_0^t\frac{\phi(t')}{\sqrt{t+\beta}}dt'}}$$

3 The exact solutions for $n=3$

In this section, we mainly consider the compressible Euler equations with $n=3$. And, we give two class of especial solutions using the system (1.4).

3.1 The first class solutions

Let

$$u^r=u_1^r(r,t),u^\theta=u_1^\theta(r,t),u^z=u_1^z(r,t),p=p_1(r,t) \quad (3.1)$$

then we obtain the one-parameter model

$$\begin{cases}(\frac{1}{r}u_1^r+u_{1r}^r)_t+u_1^r(\frac{1}{r}u_1^r+u_{1r}^r)_r+(\frac{1}{r}u_1^r+u_{1r}^r)^2=0\\ u_{1t}^r+u_1^ru_{1r}^r-\frac{1}{r}(u_1^\theta)^2+u_1^zu_{1z}^r+p_{1r}=0\\ u_{1t}^\theta+u_1^ru_{1r}^\theta+\frac{1}{r}u_1^ru_1^\theta=0\\ u_{1t}^z+u_1^ru_{1r}^z=0\end{cases} \quad (3.2)$$

Write

$$\eta=\frac{1}{r}u_1^r+u_{1r}^r$$

Then, by ODE theorem we have

$$u_1^r=\frac{e(t)}{r}+\frac{1}{r}\int r\eta dr$$

Applying the expression and (3.2), we get

$$\eta_t+(\frac{e(t)}{r}+\frac{1}{r}\int r\eta dr)\eta_r+\eta^2=0$$

Let

$$\eta(t,r)=w(t,z),z=r^2$$

we obtain

$$w_t+(2e(t)+\int wdz)w_z+w^2=0$$

Writing

$$\xi = 2e(t) + \int w dz,$$

then, in the end, we think about the problem

$$\xi_{zt} + \xi\xi_{zz} + \xi_z^2 = 0$$

By this equation, we get

$$\begin{cases} \xi_t + \xi\xi_z = h(t) \\ \xi(0,z) = g(z) \end{cases} \tag{3.3}$$

We use characteristics of the method to build solution. Hence, we have

$$\xi(t,z) = \tilde{\xi}(t,z_0) = g(z_0) + \int_0^t h(t')dt'$$

where z_0 meets

$$z = z_0 + tg(z_0) + \int_0^t \int_0^{t'} h(t'')dt''dt'$$

Choosing suitable $g(z_0)$ so that $G(z_0) = z_0 + tg(z_0) \in C^1$, then we have

$$\xi(t,z) = g\left(G^{-1}(z - \int_0^t \int_0^{t'} h(t'')dt''dt')\right) + \int_0^t h(t')dt'$$

then by the above conversions, we get

$$u_1^r(t,r) = \frac{e(t)}{r} + \frac{1}{2r} g\left(G^{-1}(r^2 - \int_0^t \int_0^{t'} h(t'')dt''dt')\right)$$

Choosing suitable $h(t), g(z_0)$, we gain the solution u_1^r. Due to the characteristics'the method, we obtain the solution u_1^θ and u_1^z depending on u_1^r. According to these works, we get the following results.

Theorem 3.1. Let α, β be constants meeting $\alpha \neq 0, \beta \neq 0$. Then the three dimension compressible Euler equations (1.2) has a class of exact solutions

$$\bar{u}(t,x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}} u_1^r - \frac{\alpha y}{x^2+y^2}, \frac{y}{\sqrt{x^2+y^2}} u^r + \frac{\alpha x}{x^2+y^2}, \beta\right), \tag{3.4}$$

$$p(t,x,y,z) = \int -u_{1t}^r dr - \beta u_1^r - \frac{\alpha^2}{2r^2} - \frac{1}{2}(u_1^r)^2, \rho(t,x,y,z) = \frac{1}{r} u_1^r + u_{1r}^r + u_{1z}^z \tag{3.5}$$

with

$$u_1^r(t,r) = \frac{e(t)}{r} + \frac{1}{2r} g\left(G^{-1}(r^2 - \int_0^t \int_0^{t'} h(t'')dt''dt')\right)$$

Here, the function $G(s) = tg(s) + s$ is any invertible function. Moreover, it also has other exact solutions

$$\bar{u}(t,x,y,z) = \left(\frac{xc(t)}{x^2+y^2} + \frac{x}{2(t+\beta)} - \frac{y}{\sqrt{x^2+y^2}} u_1^\theta, \frac{yc(t)}{x^2+y^2} + \frac{y}{2(t+\beta)} + \frac{x}{\sqrt{x^2+y^2}} u_1^\theta, u_1^z\right) \tag{3.6}$$

$$p(t,x,y,z) = \int -u^r_{1t}dr - \beta u^r_1 - \frac{\alpha^2}{2r^2} - \frac{1}{2}(u^r_1)^2, \rho(t,x,y,z) = \frac{1}{r}u^r_1 + u^r_{1r} + u^z_{1z} \tag{3.7}$$

with

$$u^\theta_1(t,r) = \frac{\psi\left((\frac{t}{\beta}+1)^{\frac{1}{4}}\sqrt{r^2 - 2\int_0^t \frac{c(t')}{\sqrt{t+\beta}}dt'}\right) r}{(\frac{t}{\beta}+1)^{\frac{1}{4}}\sqrt{r^2 - 2\int_0^t \frac{c(t')}{\sqrt{t+\beta}}dt'}}, u^z_1 = \omega\left((\frac{t}{\beta}+1)^{\frac{1}{4}}\sqrt{r^2 - 2\int_0^t \frac{c(t')}{\sqrt{t+\beta}}dt'}\right)$$

3.2 The second kind solutions

We consider the type solution

$$u^r = u^r_2(t,r), u^\theta = u^\theta_2(t,r), u^z = u^z_2(t,z), p = q_1(t,r) + q_2(t,z)$$

Then, we have the equations

$$\begin{cases} (\frac{1}{r}u^r_2 + u^r_{2r})_t + u^z_{2zt} + u^r_2(\frac{1}{r}u^r_2 + u^r_{2r})_r + u^z_2 u^z_{2zz} + (\frac{1}{r}u^r_2 + u^r_{2r})^2 + 2(\frac{1}{r}u^r_2 + u^r_{2r})u^z_{2z} + (u^z_{2z})^2 = 0 \\ u^r_{2t} + u^r_2 u^r_{2r} - \frac{1}{r}(u^\theta_2)^2 + q_{1r} = 0 \\ u^\theta_{2t} + u^r_2 u^\theta_{2r} + \frac{1}{r}u^r_2 u^\theta_2 = 0 \\ u^z_{2t} + u^z_2 u^z_{2z} + q_{2z} = 0 \end{cases} \tag{3.8}$$

Let

$$\frac{1}{r}u^r_2 + u^r_{2r} = \phi(t)$$

then, we have

$$u^r_2 = \frac{c(t)}{r} + \frac{1}{2}r\phi(t) \tag{3.9}$$

and the reduced system

$$\begin{cases} \phi' + \phi^2 = 0 \\ u^z_{2zt} + u^z_2 u^z_{2zz} + 2u^z_{2z}\phi(t) + (u^z_{2z})^2 = 0 \\ u^r_{2t} + u^r_2 u^r_{2r} - \frac{1}{r}(u^\theta_2)^2 + q_{1r} = 0 \\ u^\theta_{2t} + u^r_2 u^\theta_{2r} + \frac{1}{r}u^r_2 u^\theta_2 = 0 \\ u^z_{2t} + u^z_2 u^z_{2z} + q_{2z} = 0 \end{cases} \tag{3.10}$$

By the first equation in (3.10), we get

$$\phi(t) = \frac{1}{t+\beta}, u^r_2 = \frac{c(t)}{r} + \frac{r}{2(t+\beta)} \tag{3.11}$$

Using the second and last equation, we have

$$u^z_{2t} + u^z_2 u^z_{2z} = -2u_{2z}\phi(t)$$

with

$$q_{2z} = 2u_{2z}\phi(t)$$

Employing the characteristics of the method, we know

$$u^z_2 = \pi\left(\Pi^{-1}\left(z - 2\ln(\frac{t}{\beta}+1)\right)\right),$$

with the function Π satisfying

$$\Pi(z_0) = z_0 + t\pi(z_0) = z - 2\ln(\frac{t}{\beta} + 1)$$

As for u_2^θ, using the fourth equation and against the characteristics of the method, we gain that

$$u_2^\theta(t,r) = \frac{\psi\left((\frac{t}{\beta}+1)^{\frac{1}{4}}\sqrt{r^2 - 2\int_0^t \frac{c(t')}{\sqrt{t+\beta}}dt'}\right) r}{(\frac{t}{\beta}+1)^{\frac{1}{4}}\sqrt{r^2 - 2\int_0^t \frac{c(t')}{\sqrt{t+\beta}}dt'}}$$

Thus, we have the pressure

$$p = \int \frac{1}{r}(u_2^\theta)^2 dr - \int u_{2t}^r dr - \frac{1}{2}(u_2^r)^2 + 2\int \frac{u_{2z}}{t+\beta}dz$$

Theorem 3.2. The three dimension compressible Euler equations (1.2) has a class of exact solutions

$$\bar{u}(t,x,y,z) = \left(\frac{xc(t)}{x^2+y^2} + \frac{x}{2(t+\beta)} - \frac{y}{\sqrt{x^2+y^2}}u^\theta, \frac{yc(t)}{x^2+y^2} + \frac{y}{2(t+\beta)} + \frac{x}{\sqrt{x^2+y^2}}u^\theta, u_2^z\right) \quad (3.12)$$

$$p(t,x,y,z) = \int \frac{1}{r}(u_2^\theta)^2 dr - \int u_{2t}^r dr - \frac{1}{2}(u_2^r)^2 + 2\int \frac{u_{2z}}{t+\beta}dz, \rho(t,x,y,z) = \frac{1}{r}u_2^r + u_{2r}^r + u_{2z}^z \quad (3.13)$$

with

$$u_2^\theta(t,r) = \frac{\psi\left((\frac{t}{\beta}+1)^{\frac{1}{4}}\sqrt{r^2 - 2\int_0^t \frac{c(t')}{\sqrt{t+\beta}}dt'}\right) r}{(\frac{t}{\beta}+1)^{\frac{1}{4}}\sqrt{r^2 - 2\int_0^t \frac{c(t')}{\sqrt{t+\beta}}dt'}}, u_2^z = \pi\left(\Pi^{-1}\left(z - 2\ln(\frac{t}{\beta}+1)\right)\right)$$

Where, the function $\Pi(s) = s + t\pi(s)$ is any invertible function.

Remark 3.2.2. The two or three dimension compressible Euler equations's exact solutions in this paper, depend on the Burger's equation:

$$\begin{cases} u_t + uu_x = 0 \\ u(0,x) = \omega(x) \end{cases} \quad (3.14)$$

If $\omega(x)$ is Riemann's data, then the above equation has shock wave. Therefore, the two or three dimension compressible Euler equations has shock wave.

4 Conclusions

In this work, we utilize the system (1.3) to build up some exact solutions of the 2-dimensional and 3-dimensional compressible Euler equations. At the same time, we give some exact solutions for 3-dimensional incompressible Euler equations. However, the constructed exact solutions of incompressible system are infinite energetic, and simultaneously the blow-up solutions are also obtained via choosing certain proper variable functions.

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