Original Research Article

Certain Properties Involving the Joint Essential Numerical Ranges

Abstract

The concept of essential numerical range of an operator was defined and studied by Stampfli and Williams in 1972. Researchers generalised this idea of essential numerical range to a group of operators to the joint essential numerical range. In this paper, we consider the joint essential numerical range and show that the convexity of the classical numerical range also hold for the joint essential numerical range. Further, we show that the joint essential spectrum is contained in the joint essential numerical range by looking at the boundary of the joint essential spectrum.

Keywords: Numerical range, joint essential numerical range, joint essential spectrum.

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1 Introduction and preliminaries

Let $B(X)$ denote the algebra of (bounded) linear operators acting on complex Hilbert space X with inner product $\langle , \rangle$. The joint numerical range of an m-tuple operator $T = (T_1, ..., T_m) \in B(X)$ is denoted and defined as,

$$W_m(T) = \left\{ \Big(\langle T_1 x, x\rangle, ..., \langle T_m x, x\rangle \Big) : x \in X, \langle x, x\rangle = 1 \right\}.$$

This was studied by various researchers who sought to find out how much of the knowledge about the numerical range in the single operator case carried over to the analogous situation in the case of an m-tuple of operators. In the case $m = 1$, it is the usual numerical range of an operator T which is denoted and defined as

$$W(T) = \{\langle Tx, x\rangle \ : x \in X, \langle x, x \ \rangle = 1\}.$$

Unlike $W(T)$, the set $W_m(T)$ is generally not convex for m-tuple of operators (see [2]). However, the set $W_m(T)$ is known to be convex in the following cases;

1. $T = (T_\varphi, ..., T_\varphi)$ is an $m-$tuple of Toeplitz operators.
2. $T = (T_1, ..., T_m)$ is an $m-$tuple of commuting normal operators.
3. $T = (T_1, ..., T_m)$ is a commuting $m-$tuple of operators on a two dimensional Hilbert space.

Theorem 1.1. *If $T = (T_1, ..., T_m)$ is an $m-$tuple of commuting normal operators, then $W_m(T)$ is a convex subset of $\mathbb{C}^m$.*

See Dekker [7] for the proof.

Theorem 1.2. *Let $\varphi = (\varphi_1, ..., \varphi_m)$ be an $m-$tuple of functions in L^∞. Then $W_m(T)$ of commuting $m-$tuple $T = (T_\varphi, ..., T_\varphi)$ of Toeplitz operators on a classical Hardy space H^2 is convex.*

See Dash [5] for the proof.

We use the following theorem to show that the joint numerical range is invariant under unitary equivalence.

Theorem 1.3. *Let U be a unitary operator on X. Then $W_m(T) = W_m(U^*TU)$.*

Proof. Since U is a unitary operator, $x \in X$ is a unit vector of X if and only if U^*x is a unit vector. Note that $\langle UTU^*x, x\rangle = \langle TU^*x, U^*x\rangle = \langle Tx, x\rangle$,
Also, $\|U^*x\| = 1$ if and only if $\|x\| = 1$. The proof follows from the definition of joint numerical range. □

Related to the study of numerical range is the notion of essential numerical range for a single operator which was introduced and studied in [10] by Stampfli and Williams in 1968. It is denoted and defined as

$W_e(T) = \{r \in \mathbb{C} : \langle Tx_n, x_n\rangle \to r, x_n \to 0 \text{ weakly}\}$.

In [3], Bonsall and Duncan proved that $W_e(T)$ is nonempty, compact and satisfies
$W_e(T + \beta) = W_e(T) + \beta$ for any scalar β. Further, they showed that $0 \in W_e(T)$ if and only if T is compact. The concept of the set $W_e(T)$ was generalised to a group of operators to the joint essential numerical range, $W_{e_m}(T)$. Generalising the equivalent definitions of the essential numerical range, $W_{e_m}(T)$ is also defined as $W_{e_m}(T) = \{r_k \in \mathbb{C}^m : \langle T_k x_n, x_n\rangle \to r_k, x_n \to 0 \text{ weakly}\,; 1 \le k \le m\}$. It is related to the joint numerical range by the formula

$$W_{e_m}(T) = \bigcap \left\{ \overline{W_m(T + K)} : K = (K_1, ..., K_m) \in \mathcal{K}(X) \right\}.$$

Since $\overline{W_m(T + K)}$ is non convex [1, 2, 3, 8, 9], it is unexpected for the set $W_{e_m}(T)$ to be convex since it is an intersection of non convex sets. One of the objects of this paper is to show that the joint essential numerical range is always convex.

Many authors showed the relation between the joint numerical range and the joint spectrum. The joint spectrum $\sigma_m(T)$ of a commuting m-tuple of elements $T = (T_1, ..., T_m) \in X$ is defined as $\sigma_m(T) = \sigma_m^l(T) \cup \sigma_m^r(T)$ where the left (right) joint spectrum $\sigma_m^l(T)(\sigma_m^r(T))$ is the set of all $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{C}^m$ such that $\{b_i - \lambda_i\}_{i=1}^m$ generates a proper left (right) ideal in the Calkin algebra and $b_i = \pi(T_i)$ is the coset containing T_i $\forall i \in [1, m]$ and π the canonical homomorphism from $B(X)$ to the Calkin algebra $B(X)/\mathcal{K}(X)$. Consult Bonsall and Duncan [3] for the notion of the joint spectrum.

According to Dash [4], the joint essential spectrum $\sigma_{e_m}(T)$ of $T = (T_1, ..., T_m)$ is defined as $\sigma_{e_m}(T) = \sigma_{e_m}^l(T) \cup \sigma_{e_m}^r(T)$ where

$$\sigma_{e_m}^l(T) = \Big\{ \lambda = (\lambda_1, ..., \lambda_m) : \ B_1(T_1 - \lambda_1 I) + ... + B_m(T_m - \lambda_m I) \text{ is not a Fredholm operator for all operators } B = (B_1, ..., B_m) \text{ on } X \Big\} \text{ and}$$

$\sigma^r_{e_m}(T) = \Bigg\{ \lambda = (\lambda_1, ..., \lambda_m) : (T_1 - \lambda_1 I)B_1 + ... + (T_m - \lambda_m I)B_m$ is not a Fredholm operator for all operators $B = (B_1, ..., B_m)$ on $X \Bigg\}$.

Recall that an operator $T \in B(X)$ is said to be Fredholm if it has a closed range with finite dimensional null space and its range of finite co-dimension. We shall denote the null space and range of T by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ respectively. The index of a Fredholm operator $T \in B(X)$ is given by $i(T) = \alpha(T) - \beta(T)$ where $\alpha(T) = \dim\ (\mathcal{N}(T))$, and $\beta(T) = \text{codim}\ (\mathcal{R}(T))$.

Lemma 1.4. *(Dash [4]) Let $d = (d_1, ..., d_m)$ be an m-tuple of elements in a unital C^*-algebra of X. Then:*

(a) $(\lambda_1, ..., \lambda_m) \in \sigma^l_m(d_1, ..., d_m)$ *if and only if* $0 \in \sigma_m \sum_{i=1}^{m} (d_i - \lambda_i)^*(d_i - \lambda_i)$

(b) $(\lambda_1, ..., \lambda_m) \in \sigma^r_m(d_1, ..., d_m)$ *if and only if* $0 \in \sigma_m \sum_{i=1}^{m} (d_i - \lambda_i)(d_i - \lambda_i)^*$.

See Dash [4] for the proof.

The following proof was then used by Dash to show the relationship between the joint spectrum and the joint essential spectrum of an m-tuple of operator $T = (T_1, ..., T_m)$.

Theorem 1.5. *(Dash [5]) Let $T = (T_1, ..., T_m)$ be an $m-$tuple of operators on X. Then:*

(a) $\sigma^l_m(T) = \sigma^l_{e_m}(T) \cup \sigma_p(T)$

(b) $\sigma^r_m(T) = \sigma^r_{e_m}(T) \cup \sigma_p(T^*)^*$,
and hence we have

(c) $\sigma_m(T) = \sigma_{e_m}(T) \cup \sigma_p(T^*)^*$, *where* $T^* = (T^*_1, ..., T^*_m)$ *and star on the right represents complex conjugates.*

See Dash [5] for the proof.
Here, $\sigma_p(T)$ is a joint eigenvalue (point spectrum) of an operator $T = (T_1, ..., T_m)$ defined as a point $\lambda = (\lambda_1, ..., \lambda_m)$ such that for a nonzero eigenvector x there is $T_i x = \lambda_i x, i = (1, ..., m)$.

The following theorem by Dash [6] will be used in the sequel.

Theorem 1.6. *Let $T = (T_1, ..., T_m)$ be a commuting m-tuple operator on X. Then:*

(a) $\lambda = (\lambda_1, ..., \lambda_m) \in \sigma^l_{e_m}(T)$ *if and only if there exists a sequence* $\{x_m\}$ *of unit vectors in* X *with* $x_m \to 0$ *weakly such that* $\|(T_i - \lambda_i)x_m\| \to 0$ *as* $m \to \infty$, *for each* $i, 1 \leq i \leq m$.

(b) $\lambda = (\lambda_1, ..., \lambda_m) \in \sigma^r_{e_m}(T)$ *if and only if there exists a sequence* $\{x_m\}$ *of unit vectors in* X *with* $x_m \to 0$ *weakly such that* $\|(T^*_i - \lambda^*_i)x_m\| \to 0$ *as* $m \to \infty$, *for each* $i, 1 \leq i \leq m$.
Moreover, the sequence $\{x_m\}$ *can be chosen orthonormal.*

See Dash [6] for the proof.
This paper shows that the joint essential spectrum of $T = (T_1, ..., T_m)$ is contained in the joint essential numerical range of $T = (T_1, ..., T_m)$.

2 Joint Essential Numerical Range

In this section, we examine some of the properties of the set $W_{e_m}(T)$ defined above.

Theorem 2.1. *Let X be an infinite dimensional complex Hilbert space and $T = (T_1, ..., T_m) \in B(X)$. If $r_k = (r_1, ..., r_m) \in \mathbb{C}^m$ then there exists an orthonormal sequence $\{x_n\} \in X$ such that $\langle T_k x_n, x_n\rangle \to r_k; 1 \leq k \leq m$ if and only if $r_k \in W_{e_m}(T)$.*

Proof. Suppose that for a point $r_k = (r_1, ..., r_m) \in \mathbb{C}^m$ there exists an orthonormal sequence $\{x_n\} \in X$ such that $\langle T_k x_n, x_n\rangle \to r_k; 1 \leq k \leq m$. Since every orthonormal sequence $\{x_n\}$ converges weakly to zero and $\|x_n\| = 1$, we have that $r_k \in W_{e_m}(T)$.

Conversely, let $r_k = (r_1, ..., r_m) \in W_{e_m}(T)$ and show that there exists an orthonormal sequence $\{x_n\} \in X$ such that $\langle T_k x_n, x_n\rangle \to r_k; 1 \leq k \leq m$. Suppose $r_k \in W_e(\widetilde{T})$. Then there is a sequence $\{x_n\}$ of vectors such that $\langle T_k x_n, x_n\rangle \to r_k,\ \|x_n\| = 1, x_n \to 0$ weakly. Choosing the set $\{x_1, ..., x_n\}$ which satisfy $|\langle T_k x_n, x_n\rangle - r| < \frac{1}{i}\ \forall i$ and letting $\mathcal{M}$ be the subspace spanned by $x_1, ..., x_n$ and P be the projection onto $\mathcal{M}$ then we have $\|Px_n\| \to 0$ as $n \to \infty$. Suppose $z_n = \|(I-P)x_n\|^{-1}((I-P)x_n)$. We obtain $T_k z_n = \left\|(I-P)x_n\right\|^{-1}\Big(T_k(I-P)x_n\Big)$. This gives

$$\begin{aligned}\langle T_k z_n, z_n\rangle &= \left\langle \left\|(I-P)x_n\right\|^{-1}\Big(T_k(I-P)x_n\Big), \left\|(I-P)x_n\right\|^{-1}\Big(T_k(I-P)x_n\Big)\right\rangle \\ &= \left\|(I-P)x_n\right\|^{-2}\Big\{\langle T_k x_n, x_n\rangle - \langle T_k x_n, Px_n\rangle - \langle T_k Px_n, x_n\rangle + \langle T_k Px_n, Px_n\rangle\Big\} \\ &\to r_k.\end{aligned}$$

We choose n large enough such that $|\langle T_k z_n, z_n\rangle - r_k| < \frac{1}{n+1}$.
If we let $z_n = x_{n+1}$ we get $|\langle T_k x_{n+1}, x_{n+1}\rangle - r_k| < \frac{1}{n+1}$ which completes the proof. □

Before we prove the following result, we remind the reader that a subset $\mathcal{A}$ of a linear space M is convex if $\forall x, y \in \mathcal{A}$ the segment joining x and y is contained in $\mathcal{A}$, that is, $\lambda x + (1-\lambda)y \in \mathcal{A}\ \forall \lambda \in [0, 1]$. A set S is starshaped if $\exists y \in S$ such that $\forall x \in S$ the segment joining x and y is contained in S, that is $\lambda x + (1-\lambda)y \in S\ \forall \lambda \in [0, 1]$. A point $y \in S$ is a star center of S if there is a point $x \in S$ such that the segment joining x and y is contained in S. A convex set is starshaped with all its points being star centers.

Theorem 2.2. *Suppose $T = (T_1, ..., T_m) \in B(X)$. Then $W_{e_m}(T)$ is nonempty, compact and each element $r_k \in W_{e_m}(T)$ is a star center of $\overline{W_m(T)}$. Moreover, $W_{e_m}(T)$ is convex.*

Proof. First, we prove that $W_{e_m}(T)$ is nonempty. To do this, from Theorem 2.1, there exists an orthonormal sequence $\{x_n\} \in X$ such that $\langle T_k x_n, x_n\rangle \to r_k; 1 \leq k \leq m$. Thus the sequence $\{\langle T_k x_n, x_n\rangle\}_{n=1}^{\infty}$ is bounded. Choose a subsequence and assume that $\langle T_k x_n, x_n\rangle$ converges. Then $W_{e_m}(T)$ is nonempty.

The compactness of $W_{e_m}(T)$ can be seen right from its definition. That is, the joint essential numerical range is defined as the intersection of all sets of the form $\overline{W_m(T+K)} : K = (K_1, ..., K_m)$ where $\mathcal{K}(X)$ denote the sets of compact operators in $B(X)$. Being an intersection of compact sets, the joint essential numerical range is also compact.

To prove that each element $r_k \in W_{e_m}(T)$ is a star center of $\overline{W_m(T)}$, it should be shown that $(1-\lambda)p + \lambda r_k \in \overline{W_m(T)} : \lambda \in [0, 1]$ where $r_k \in W_{e_m}(T)$ and $p \in \overline{W_m(T)}$. Assume without loss of generality that $\|T\| = 1$. Suppose $s \in \overline{W_m(T)}$ so that $s = \lambda r_k + (1-\lambda)p$. Let $\{x_n\}$ and $\{e_n\}$

be orthonormal sequences in X such that $r_k = \langle Tx_n, x_n\rangle, p = \langle Te_n, e_n\rangle$ and $\|x_n\| = \|e_n\| = 1$. Then,

$$\begin{aligned} s &= \lambda\langle Tx_n, x_n\rangle + (1-\lambda)\langle Te_n, e_n\rangle \\ &= \left\langle T\sqrt{\lambda}\, x_n, \sqrt{\lambda}\, x_n \right\rangle + \left\langle T\sqrt{1-\lambda}\, e_n, \sqrt{1-\lambda}\, e_n \right\rangle \\ &= \left\langle (T\sqrt{\lambda}\, x_n + T\sqrt{1-\lambda}\, e_n), (\sqrt{\lambda}\, x_n + \sqrt{1-\lambda}\, e_n) \right\rangle \\ \left\| \sqrt{\lambda}\, x_n + \sqrt{1-\lambda}\, e_n \right\|^2 &= \left(\left\| \sqrt{\lambda}\, x_n \right\|^2 + \left\| \sqrt{1-\lambda}\, e_n \right\|^2 \right) \\ &= \lambda \parallel x_n \parallel^2 + (1-\lambda) \parallel e_n \parallel^2 \\ &= \lambda + (1-\lambda) = 1 \end{aligned}$$

Thus, $(1-\lambda)r_k + \lambda p \in \overline{W_m(T)}$.

Convexity of $W_{e_m}(T)$ is proved by showing that for $r_k, p \in W_{e_m}(T)$ and $\lambda \in [0,1]$ we have $\lambda r_k + (1-\lambda)p \in W_{e_m}(T)$. Now, $r_k \in W_{e_m}(T) = W_{e_m}(T+K)$ for every $K \in \mathcal{K}(X)$ and $p \in W_{e_m}(T) = W_{e_m}(T+K) \subseteq \overline{W_m(T+K)}$. By Theorem 2.2, $\lambda r_k + (1-\lambda)p \in \overline{W_m(T+K)}$. Thus, $\lambda r_k + (1-\lambda)p \in \cap\{\overline{W_m(T+K)} : K \in \mathcal{K}(X)\} = W_{e_m}(T)$. Hence $W_{e_m}(T)$ is convex. □

The following theorem shows that the joint essential spectrum is contained in the joint essential numerical range. We use Theorem 1.6 to arrive at our result.

Theorem 2.3. *Let X be an infinite dimensional complex Hilbert space and $T = (T_1, ..., T_m) \in B(X)$. Then $\sigma_{e_m}(T) \subseteq W_{e_m}(T)$.*

Proof. Let $\lambda = (\lambda_1, ..., \lambda_m) \in \sigma_{e_m}(T)$. It should be shown that $\lambda = (\lambda_1, ..., \lambda_m) \in W_{e_m}(T)$. To do this, since $\sigma_{e_m}(T) = \sigma^l_{e_m}(T) \cup \sigma^r_{e_m}(T)$, it is enough to show that both $\sigma^l_{e_m}(T)$ and $\sigma^r_{e_m}(T)$ are contained in $W_{e_m}(T)$. Now suppose $\lambda = (\lambda_1, ..., \lambda_m) \in \sigma^l_{e_m}(T)$. Then there is a sequence $\{x_m\}$ of unit vectors in X such that $\|(T_i - \lambda_i I)x_m\| \to 0\ \forall i = (1, ..., m)$ as $x_m \to 0$ weakly.
Now $|\langle (T_i - \lambda_i I)x_m, x_m\rangle| \le \|(T_i - \lambda_i I)x_m\| \to 0\ \ \forall i = (1, ..., m)$.
Therefore, $\langle T_i x_m, x_m\rangle \to \lambda_i\ \forall i = (1, ..., m)$. Thus $\lambda = (\lambda_1, ..., \lambda_m) \in W_{e_m}(T)$.
Likewise, let $\lambda = (\lambda_1, ..., \lambda_m) \in \sigma^r_{e_m}(T)$ then $\lambda^* = (\lambda^*_1, ..., \lambda^*_m) \in \sigma^l_e(T)^*$.
This gives $\lambda = (\lambda_1, ..., \lambda_m) \in W_{e_m}(T)^* = [W_{e_m}(T)]^*$ (the complex conjugate of $W_{e_m}(T)$) implying that $\lambda = (\lambda_1, ..., \lambda_m) \in W_{e_m}(T)$. □

3 Conclusions

In section (2), equivalent definitions of the joint essential numerical range were proved. We also proved that the set $W_{e_m}(T)$ in nonempty, compact and convex. Further, it was shown that the joint essential spectrum of an m-tuple operator $T = (T_1, ..., T_m) \in B(X)$ is contained in the joint essential numerical range of an m-tuple operator $T = (T_1, ..., T_m) \in B(X)$.

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