

Original Research Article

On second order slope rotatable designs under intra-class correlated structure of errors using a pair of dissimilar incomplete block designs

Abstract: In this paper, a study of second order slope rotatable designs under intra-class correlation error structure using two suitably chosen dissimilar incomplete block designs like balanced incomplete block designs and symmetrical unequal block arrangements with two unequal block sizes are suggested. Further, we study the variance of the estimated slopes for different values of the intra-class correlation coefficient (ρ) and the distance from the centre (d) for v factors are suggested. Some illustrative examples are also suggested.

Keywords: Second order response surface designs, intra-class correlated errors.

1. Introduction

In the context of response surface methodology, Box and Hunter [1] introduced the concept of rotatability assuming errors are uncorrelated and homoscedastic. Das and Narasimham [2] constructed rotatable designs through balanced incomplete block designs (BIBD). Narasimham et al. [8] constructed rotatable designs through a pair of incomplete block designs (IBD). Hader and Park [7] constructed slope rotatable central composite designs. Victorbabu and Narasimham [19, 20] constructed second order slope rotatable designs (SOSRD) through BIBD and a pair of IBD respectively. Victorbabu [17, 18] constructed SOSRD using symmetrical unequal block arrangements (SUBA) with two unequal block sizes and suggested a review on SOSRD respectively. Victorbabu et al. [21] studied SOSRD using a pair of dissimilar IBD.

So far all the authors studied rotatable and slope rotatable designs assuming errors to be uncorrelated and homoscedastic. However, it is not uncommon to come across the practical situations when errors are correlated, violating usual assumptions. Das [3, 4, 5] introduced and studied robust second order rotatable designs (RSORD). Das [6] studied slope rotatability with correlated errors. Rajyalashmi [10] constructed second order rotatable and slope rotatable designs under different correlated error structures, Rajyalakshmi and Victorbabu [11, 12, 13] developed SOSRD under intra-class correlation structure of errors

using central composite designs, symmetrical unequal block arrangements (SUBA) with two unequal block sizes and BIBD respectively. Sulochana and Victorbabu [14, 15, 16] studied SOSRD under intra-class correlated structure of errors using a pair of BIBD, SUBA with two unequal block sizes and partially **balanced incomplete block type designs**.

In this paper following the works of Das [6], Rajyalakshmi and Victorbabu [11, 12, 13] a study on SOSRD under intra-class correlation error structure using a pair of dissimilar incomplete block designs are suggested. Further, we study the variance function of the estimated slopes for different values of intra-class correlated coefficient (ρ) and also obtain the distance from centre (d) for 'v' (v number of factors).

2. Conditions for SOSRD under intra-class correlated structure of errors (cf. Das [6], Rajyalakshmi and Victorbabu [11, 12, 13])

A second order response surface design $D = \left((X_{iu}) \right)$ for fitting,

$$y_u(x) = b_0 + \sum_{i=1}^v b_i x_{iu} + \sum_{i=1}^v b_{ii} x_{iu}^2 + \sum_{i=1}^v \sum_{i < j}^v b_{ij} x_{iu} x_{ju} + \varepsilon_u \quad (2.1)$$

where x_{iu} denotes the level of the i^{th} factor ($i=1,2,\dots,v$) in the u^{th} run ($u=1,2,\dots,N$) of the experiment, ε_u 's are correlated random errors, is said to be a SOSRD under intra-class correlated structure of errors, if the variance of the estimate of first order partial derivative of $y_u(x_{1u}, x_{2u}, x_{3u}, \dots, x_{vu})$ with respect to each independent variable X_i is only a function of the distance $\left(d^2 = \sum_{i=1}^v X_i^2 \right)$ of the point $(x_{1u}, x_{2u}, x_{3u}, \dots, x_{vu})$ from the origin (centre of the design). Such a spherical variance function for estimation of slopes in the second order response surface is achieved if the design points satisfy the following conditions (cf. (Das [6], Rajyalakshmi and Victorbabu [11, 12, 13]).

The necessary and sufficient conditions for SOSRD under intra-class correlation error structure are

$$\begin{aligned} \sum_{u=1}^N x_{iu} = 0, \sum_{u=1}^N x_{iu} x_{ju} = 0, \sum_{u=1}^N x_{iu} x_{ju}^2 = 0, \sum_{u=1}^N x_{iu} x_{ju} x_{ku} = 0, \sum_{u=1}^N x_{iu}^3 = 0, \\ \sum_{u=1}^N x_{iu} x_{ju}^3 = 0, \sum_{u=1}^N x_{iu} x_{ju} x_{ku}^2 = 0, \sum_{u=1}^N x_{iu} x_{ju} x_{ku} x_{lu} = 0, \text{ for } i \neq j \neq k \neq l; \end{aligned} \quad (2.2)$$

$$\sum_{u=1}^N x_{iu}^2 = \text{constant} = N\theta_2 \quad (2.3)$$

$$\sum_{u=1}^N x_{iu}^4 = \text{constant} = cN\theta_4, \text{ for all } i \quad (2.4)$$

$$\sum_{u=1}^N x_{iu}^2 x_{ju}^2 = \text{constant} = N\theta_4, \text{ for all values } i \neq j \quad (2.5)$$

From (2.4) and (2.5), we have,

$$\sum_{u=1}^N x_{iu}^4 = c \sum_{u=1}^N x_{iu}^2 x_{ju}^2 \quad (2.6)$$

where c , θ_2 and θ_4 are constants. The summation is over the designs points.

Using the above simple symmetric conditions, the variances and covariances of the estimated parameters under the intra-class correlation error structure are as follows:

$$V(\hat{b}_0) = \frac{[\theta_4(c+v-1)A - v\rho N\theta_2^2]\sigma^2 A}{N\Delta} \quad (2.7)$$

$$V(\hat{b}_i) = \frac{\sigma^2(1-\rho)}{N\theta_2} \quad (2.8)$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2(1-\rho)}{N\theta_4} \quad (2.9)$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2(1-\rho)\theta_4[(c+v-2)A - (v-1)\rho N\theta_2^2 - (v-1)\theta_2^2(1-\rho)]}{(c-1)N\theta_4\Delta} \quad (2.10)$$

$$\text{Cov}(\hat{b}_0, \hat{b}_{ii}) = \frac{-\sigma^2\theta_2^2(1-\rho)A}{N\Delta} \quad (2.11)$$

$$\text{Cov}(\hat{b}_{ii}, \hat{b}_{ij}) = \frac{\sigma^2(1-\rho)[\theta_2^2(1-\rho) - \theta_4A + \rho N\theta_2^2]}{(c-1)N\theta_4\Delta} \quad (2.12)$$

Where $A = \{1 + (N-1)\rho\}$, $\Delta = \left[\theta_4 (c+v-1) A - v\rho N \theta_2^2 - v\theta_2^2 (1-\rho) \right]$,

and the other covariances are zero.

An inspection of the variance of \hat{b}_0 shows that a necessary condition for the existence of a non-singular SOSRD under intra-class correlation error structure is

$$\theta_4 (c+v-1) A - v\rho N \theta_2^2 - v\theta_2^2 (1-\rho) > 0 \quad (2.13)$$

From (2.13), we have,

$$\frac{\theta_4}{\theta_2^2} > \frac{v}{c+v-1} \quad (\text{non-singularity condition}) \quad (2.14)$$

If the non-singularity condition (2.14) exists then only the design exists.

For the second order model,

$$\begin{aligned} \frac{\partial \hat{y}_u}{\partial x_i} &= \hat{b}_i + 2\hat{b}_{ii} x_i + \sum_{i=1, j \neq i}^v \hat{b}_{ij} x_j, \\ V \left(\frac{\partial \hat{y}_u}{\partial x_i} \right) &= V \left(\hat{b}_i \right) + 4x_i^2 V \left(\hat{b}_{ii} \right) + \sum_{i=1, j \neq i}^v x_j^2 V \left(\hat{b}_{ij} \right) \end{aligned} \quad (2.15)$$

The condition for right hand side of equation (2.15) to be a function of $(d^2 = \sum_{i=1}^v X_i^2)$ alone (for slope rotatability) is clearly,

$$V \left(\hat{b}_{ii} \right) = \frac{1}{4} V \left(\hat{b}_{ij} \right) \quad (2.16)$$

(cf. Hader and Park [7])

On simplification of (2.16) using (2.9) and (2.10) leads to

$$\left(\frac{AcN\theta_4-B}{1-\rho}\right)\left[4N-\left(\frac{AcN\theta_4-B}{A\theta_4}\right)v\left(\frac{N\theta_2^2(1-\rho)}{A\theta_4}\right)-(v-2)\left(\frac{AN\theta_4-B}{A\theta_4}\right)\right]+$$

$$\left(\frac{AN\theta_4-B}{1-\rho}\right)\left[4(v-2)-(v-1)\left(\frac{AN\theta_4-B}{AN\theta_4}\right)\right]-N^2\theta_2^2\left[4(v-1)+v\left(\frac{AN\theta_4-B}{AN\theta_4}\right)\right]=0 \quad (2.17)$$

$$A=\{1+(N-1)\rho\}, B=\rho N^2\theta_2^2$$

On simplification of (2.17) we get

$$\left(\{1+(N-1)\rho\}^2\right)\left[\theta_4\left[v(5-c)-(c-3)^2\right]+\theta_2^2\left[v(c-5)+4\right]\right]=0 \quad (2.18)$$

If $\rho = 0$ condition (2.18) equal to

$$\theta_4\left[v(5-c)-(c-3)^2\right]+\theta_2^2\left[v(c-5)+4\right]=0 \quad (2.19)$$

This is similar to the SOSRD condition of Victorbabu and Narasimham [20].

Therefore, equations (2.2) to (2.12), (2.14) to (2.18) give a set of conditions for SOSRD under intra-class correlation error structure for any general second order response surface design. Further,

$$V\left(\frac{\hat{\partial y_u}}{\partial x_i}\right)=\frac{1}{N}\left(\frac{1}{\theta_2}+\frac{d^2}{\theta_4}\right)(1-\rho)\sigma^2 \quad (2.20)$$

3. Construction of SOSRD under intra-class correlated structure of errors using a pair of dissimilar incomplete block designs

Following the methods of construction of Das [6], Victorbabu and Narasimham [19, 20], Victorbabu [18], Victorbabu et.al [21] Rajyalakshmi and Victorbabu [11, 12, 13], here a study on SOSRD with intra-class correlation error structure using two suitably chosen dissimilar incomplete block designs like BIBD and SUBA with two unequal block sizes are studied. Let ρ be the correlation errors of any two observations, each having the same σ^2 . In this paper, we follow the notations of Das and Narasimham [2], Narasimham et al. [8], and Victorbabu and Narasimham [19, 20].

Let $D_1 = (v, b_1, r_1, k_1, \lambda_1)$ denote a BIBD with $r_1 \leq c\lambda_1$ and $D_2 = (v, b_2, r_2, k_{21}, k_{22}, b_{21}, b_{22}, \lambda_2)$ denote a SUBA with two unequal block sizes (where b_{21} blocks each of size k_{21} and b_{22} blocks each of size k_{22}) with $b_{21} + b_{22} = b_2$, $k_2 = \max(k_{21}, k_{22})$ and $r_2 \geq c\lambda_2$ in v -treatments respectively. Let $2^{t(k_1)}$ and $2^{t(k_2)}$ denote fractional replicate of 2^{k_1} and 2^{k_2} with levels ± 1 levels in which no interaction with less than five factors is confounded. $[1 - (v, b_1, r_1, k_1, \lambda_1)] 2^{t(k_1)}$ denote the design points generated from the transpose of incidence matrix of the design D_1 by “multiplication” (Raghavarao, (1971), pp. 298-300). Let $[1 - (v, b_1, r_1, k_1, \lambda_1)] 2^{t(k_1)}$ are the $b_1 2^{t(k_1)}$ design points generated from D_1 by “multiplication” $[\alpha - (v, b_2, r_2, k_{21}, k_{22}, b_{21}, b_{22}, \lambda_2)] 2^{t(k_2)}$ are the $b_2 2^{t(k_2)}$ design points generated from D_2 by ‘multiplication’. Let n_0 be the number of central points in the design. The method of construction of SOSRD under intra-class correlation error structure is given in the following theorem.

Theorem 3.1:

Let $D_1 = (v, b_1, r_1, k_1, \lambda_1)$ and $D_2 = (v, b_2, r_2, k_{21}, k_{22}, b_{21}, b_{22}, \lambda_2)$ are two dissimilar incomplete block designs, $2^{t(k_1)}$ and $2^{t(k_2)}$ factorials with levels ± 1 and n_0 is pre-fixed points, then the design points,

$[1 - (v, b_1, r_1, k_1, \lambda_1)] 2^{t(k_1)} \cup [\alpha - (v, b_2, r_2, k_{21}, k_{22}, b_{21}, b_{22}, \lambda_2)] 2^{t(k_2)} \cup n_0$ give a v -dimensional SOSRD under intra-class correlation error structure in $N = b_1 2^{t(k_1)} + b_2 2^{t(k_2)} + n_0$ design points, where α^2 is a positive real root of the biquadratic equation,

$$\begin{aligned}
& \left[2^{2t(k_2)} N \{ r_2 \lambda_2 (6-v) - r_2^2 + \lambda_2^2 (5v-9) \} + 2^{3t(k_2)} r_2^2 (vr_2 - 5v\lambda_2 + 4\lambda_2) \right] A^2 \alpha^8 + \\
& \left[2^{t(k_1)+2t(k_2)+1} r_1 r_2 (vr_2 - 5v\lambda_2 + 4\lambda_2) \right] A^2 \alpha^6 + \\
& \left[2^{t(k_1)+t(k_2)} N \{ \lambda_1 \lambda_2 (10v-18) + (6-v)(r_1 \lambda_2 + r_2 \lambda_1) - 2r_1 r_2 \} + \right. \\
& \left. 2^{t(k_1)+2t(k_2)} r_2^2 (vr_1 - 5v\lambda_1 + 4\lambda_1) + 2^{2t(k_1)+t(k_2)} r_1^2 (vr_2 - 5v\lambda_2 + 4\lambda_2) \right] A^2 \alpha^4 + \\
& \left[2^{2t(k_1)+t(k_2)+1} r_1 r_2 (vr_1 - 5v\lambda_1 + 4\lambda_1) \right] A^2 \alpha^2 + \\
& \left[2^{2t(k_1)} N \{ \lambda_1^2 (5v-9) + r_1 \lambda_1 (6-v) - r_1^2 \} + 2^{3t(k_1)} r_1^2 (vr_1 - 5v\lambda_1 + 4\lambda_1) \right] A^2 = 0
\end{aligned}$$

where $A = \{1 + (N-1)\rho\}$

(3.1)

Proof: For the design points generated from a pair of dissimilar IBD, simply symmetry conditions (2.2) are true. Further, from (2.3), (2.4) and (2.5) we have,

$$\sum_{u=1}^N x_{iu}^2 = r_1^2 2^{t(k_1)} + r_2^2 2^{t(k_2)} \alpha^2 = N\theta_2, \text{ for all } i \quad (3.2)$$

$$\sum_{u=1}^N x_{iu}^4 = r_1^4 2^{t(k_1)} + r_2^4 2^{t(k_2)} \alpha^4 = cN\theta_4, \text{ for all } i \quad (3.3)$$

$$\sum_{u=1}^N x_{iu}^2 x_{ju}^2 = \lambda_1^2 2^{t(k_1)} + \lambda_2^2 2^{t(k_2)} \alpha^4 = N\theta_4, \text{ for all } i \neq j \quad (3.4)$$

Substituting θ_2 , θ_4 and c in condition (2.18) and on simplification, we get (3.1). The design exists only if at least one positive real root exists for equation (3.1). Solving equation (3.1) we get the SOSRD with intra-class correlation error structure using a pair of dissimilar IBD with different ' α ' values for the ' v ' different factors. The variance of estimated slopes of these SOSRD for $0 \leq \rho \leq 0.9$ and for $12 \leq v \leq 16$ factors are given in Table 1.

Example: We illustrate the above the method with construction of SOSRD with intra-class correlation error structure for $v=12$ -factors with the help of a pair of dissimilar incomplete block designs with parameters

$$D_1 = (v=12, b_1=44, r_1=11, k_1=3, \lambda_1=2),$$

$$D_2 = (v=12, b_2=26, r_2=6, k_{21}=2, k_{22}=3, b_{21}=6, b_{22}=20, \lambda_2=1) \text{ is given below.}$$

The design points,

$$\left[1 - \left(v=12, b_1=44, r_1=11, k_1=3, \lambda_1=2\right)\right] 2^3 U$$

$$\left[\alpha - \left(v=12, b_2=26, r_2=6, k_{21}=2, k_{22}=3, b_{21}=6, b_{22}=20, \lambda_2=1\right)\right] 2^3 U n_0$$

will give SOSRD under intra-class correlation error structure in $N = 561$ design points for 12 factors. We have,

$$\sum_{u=1}^N x_{iu}^2 = 88 + 48\alpha^2 = N\theta_2 \quad (3.5)$$

$$\sum_{u=1}^N x_{iu}^4 = 88 + 48\alpha^4 = cN\theta_4 \quad (3.6)$$

$$\sum_{u=1}^N x_{iu}^2 x_{ju}^2 = 16 + 8\alpha^4 = N\theta_4 \quad (3.7)$$

From (3.6) and (3.7), we get $c = \frac{88 + 48\alpha^4}{16 + 8\alpha^4}$. Substituting for θ_2, θ_4 and c in (2.18) and on simplification, we get the following biquadratic equation in α^2 .

$$(1 + 560\rho)^2 \left(459072\alpha^8 - 1081344\alpha^6 + 1009792\alpha^4 - 1351680\alpha^2 + 520256\right) = 0 \quad (3.8)$$

Equation (3.8) has only one positive real root $\alpha^2 = 1.8478$ ($(\forall(0 < \rho < 1))$). This can be alternatively written directly from equation (3.1). Solving (3.8), we get $\alpha = 1.3593 \forall \rho$. Substituting this value of ' α ' in (3.5), (3.6) and (3.7) we obtain $\theta_2 = 0.3149, \theta_4 = 0.0772, c = 5.5183$. It can be verified that non-singularity condition (2.14) is also satisfied. From (2.7) to (2.12), we can obtain the variances and covariances. Further from (2.20), we have,

$$V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right) = (0.0057 + 0.0230d^2)(1 - \rho)\sigma^2 \quad (3.9)$$

4. A study on dependence of the variance function of the response at different design points:

Here, we study the dependence of variance function of response at different designs points of SOSRD under intra-class correlation error structure using a pair of dissimilar incomplete block designs. Given 'v' factors different values of intra-class correlation coefficient ' ρ ', and distance from centre 'd' between 0 and 1, the variances are tabulated.

From (3.9) the variance of the estimated derivative is obtained by

$$V\left(\frac{\hat{\partial y_u}}{\partial x_i}\right)=0.0053(\text{taking } \rho=0.1, d=0.1, \sigma=1)$$

The numerical calculations are provided in Table 1 and Table 2.

Conclusions: From Table 1 and Table 2, we conclude that,

(i) For given v and ρ , $V\left(\frac{\hat{\partial y_u}}{\partial x_i}\right)$ increases as d increases.

(ii) For given v and d, $V\left(\frac{\hat{\partial y_u}}{\partial x_i}\right)$ decreases as ρ increases

Table 1: The variances of estimated derivatives (slopes) for the $12 \leq v \leq 16$ factors

ρ	(v=12, $b_1=44$, $r_1=11$, $k_1=3$, $\lambda_1=2$) (v=12, $b_2=26$, $r_1=6$, $k_{11}=2$, $k_{12}=3$, $b_{11}=6$, $b_{12}=20$, $\lambda_2=1$) $\alpha=1.8822$, N=561	(v=12, $b_1=44$, $r_1=11$, $k_1=3$, $\lambda_1=2$) (v=12, $b_2=26$, $r_1=6$, $k_{11}=2$, $k_{12}=3$, $b_{11}=6$, $b_{12}=20$, $\lambda_2=1$) $\alpha=1.3593$, N=737
	$V\left(\frac{\hat{\partial y_u}}{\partial x_i}\right)$	$V\left(\frac{\hat{\partial y_u}}{\partial x_i}\right)$
0	$0.0057\sigma^2 + 0.0231d^2\sigma^2$	$0.0029\sigma^2 + 0.0067d^2\sigma^2$
0.1	$0.0051\sigma^2 + 0.0208d^2\sigma^2$	$0.0026\sigma^2 + 0.0061d^2\sigma^2$
0.2	$0.0045\sigma^2 + 0.0185d^2\sigma^2$	$0.0023\sigma^2 + 0.0054d^2\sigma^2$
0.3	$0.0039\sigma^2 + 0.0161d^2\sigma^2$	$0.0020\sigma^2 + 0.0047d^2\sigma^2$
0.4	$0.0034\sigma^2 + 0.0139d^2\sigma^2$	$0.0017\sigma^2 + 0.0040d^2\sigma^2$
0.5	$0.0028\sigma^2 + 0.0115d^2\sigma^2$	$0.0014\sigma^2 + 0.0033d^2\sigma^2$
0.6	$0.0023\sigma^2 + 0.0062d^2\sigma^2$	$0.0012\sigma^2 + 0.0027d^2\sigma^2$
0.7	$0.0017\sigma^2 + 0.0069d^2\sigma^2$	$0.0008\sigma^2 + 0.0020d^2\sigma^2$
0.8	$0.0011\sigma^2 + 0.0046d^2\sigma^2$	$0.0005\sigma^2 + 0.0013d^2\sigma^2$
0.9	$0.0005\sigma^2 + 0.0023d^2\sigma^2$	$0.0003\sigma^2 + 0.0007d^2\sigma^2$

	($v=12, b_1=44, r_1=11, k_1=6, \lambda_1=5$) ($v=12, b_2=26, r_1=6, k_{11}=2, k_{12}=3, b_{11}=6, b_{12}=20, \lambda_2=1$), $\alpha=2.6815, N=913$	($v=16, b_1=48, r_1=15, k_1=5, \lambda_1=4$) ($v=16, b_2=28, r_1=6, k_2=4, b_{11}=12, b_{12}=16, \lambda_2=1$), $\alpha=2.6124, N=1217$
ρ	$V \left(\frac{\hat{\partial y_u}}{\partial x_i} \right)$	$V \left(\frac{\hat{\partial y_u}}{\partial x_i} \right)$
0	$0.0014 \sigma^2 + 0.0017 d^2 \sigma^2$	$0.0011 \sigma^2 + 0.0012 d^2 \sigma^2$
0.1	$0.0013 \sigma^2 + 0.0016 d^2 \sigma^2$	$0.0010 \sigma^2 + 0.0011 d^2 \sigma^2$
0.2	$0.0011 \sigma^2 + 0.0013 d^2 \sigma^2$	$0.0009 \sigma^2 + 0.0009 d^2 \sigma^2$
0.3	$0.0010 \sigma^2 + 0.0012 d^2 \sigma^2$	$0.0008 \sigma^2 + 0.0009 d^2 \sigma^2$
0.4	$0.0009 \sigma^2 + 0.0010 d^2 \sigma^2$	$0.0007 \sigma^2 + 0.0008 d^2 \sigma^2$
0.5	$0.0007 \sigma^2 + 0.0009 d^2 \sigma^2$	$0.0006 \sigma^2 + 0.0006 d^2 \sigma^2$
0.6	$0.0006 \sigma^2 + 0.0007 d^2 \sigma^2$	$0.0004 \sigma^2 + 0.0005 d^2 \sigma^2$
0.7	$0.0004 \sigma^2 + 0.0005 d^2 \sigma^2$	$0.0004 \sigma^2 + 0.0004 d^2 \sigma^2$
0.8	$0.0003 \sigma^2 + 0.0003 d^2 \sigma^2$	$0.0002 \sigma^2 + 0.0002 d^2 \sigma^2$
0.9	$0.0001 \sigma^2 + 0.0002 d^2 \sigma^2$	$0.0001 \sigma^2 + 0.0001 d^2 \sigma^2$

	($v=16, b_1=, r_1=6, k_1=6, \lambda_1=2$) ($v=16, b_2=28, r_1=6, k_2=4, b_{11}=12, b_{12}=16, \lambda_2=1$), $\alpha=1.8394, N=961$	($v=16, b_1=48, r_1=1, k_1=5, \lambda_1=4$) ($v=16, b_2=36, r_1=7, k_{11}=4, k_{12}=3, b_{11}=4, b_{12}=32, \lambda_2=1$), $\alpha=1.5929, N=1345$
ρ	$V \left(\frac{\hat{\partial y_u}}{\partial x_i} \right)$	$V \left(\frac{\hat{\partial y_u}}{\partial x_i} \right)$
0	$0.0019 \sigma^2 + 0.0040 d^2 \sigma^2$	$0.0013 \sigma^2 + 0.0043 d^2 \sigma^2$
0.1	$0.0017 \sigma^2 + 0.0036 d^2 \sigma^2$	$0.0011 \sigma^2 + 0.0039 d^2 \sigma^2$
0.2	$0.0015 \sigma^2 + 0.0032 d^2 \sigma^2$	$0.0010 \sigma^2 + 0.0035 d^2 \sigma^2$
0.3	$0.0014 \sigma^2 + 0.0028 d^2 \sigma^2$	$0.0009 \sigma^2 + 0.0030 d^2 \sigma^2$
0.4	$0.0011 \sigma^2 + 0.0024 d^2 \sigma^2$	$0.0008 \sigma^2 + 0.0026 d^2 \sigma^2$
0.5	$0.0009 \sigma^2 + 0.0020 d^2 \sigma^2$	$0.0007 \sigma^2 + 0.0022 d^2 \sigma^2$
0.6	$0.0008 \sigma^2 + 0.0016 d^2 \sigma^2$	$0.0005 \sigma^2 + 0.0017 d^2 \sigma^2$
0.7	$0.0006 \sigma^2 + 0.0012 d^2 \sigma^2$	$0.0004 \sigma^2 + 0.0013 d^2 \sigma^2$
0.8	$0.0004 \sigma^2 + 0.0008 d^2 \sigma^2$	$0.0003 \sigma^2 + 0.0009 d^2 \sigma^2$
0.9	$0.0001 \sigma^2 + 0.0004 d^2 \sigma^2$	$0.0001 \sigma^2 + 0.0004 d^2 \sigma^2$

Table 2: Study of dependence of estimated slope of second order response surface design under intra-class correlation error structure using a pair of dissimilar incomplete block designs at different design points for $12 \leq v \leq 16$ factors for different values of ' ρ ', d and $\sigma=1$.

((12, 44, 11, 3, 2), (12, 26, 6, 2, 3, 6, 20, 1)), a=1.3593, N=561										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0059	0.0066	0.0077	0.0094	0.0114	0.0139	0.0169	0.0204	0.0244	0.0287
0.1	0.0053	0.0059	0.0069	0.0085	0.0102	0.0126	0.0153	0.0183	0.0219	0.0259
0.2	0.0047	0.0053	0.0062	0.0075	0.0091	0.0112	0.0136	0.0163	0.0195	0.0229
0.3	0.0041	0.0046	0.0054	0.0065	0.0080	0.0098	0.0119	0.0143	0.0171	0.0201
0.4	0.0035	0.0039	0.0046	0.0056	0.0069	0.0084	0.0102	0.0122	0.0146	0.0172
0.5	0.0029	0.0033	0.0039	0.0047	0.0057	0.0069	0.0085	0.0102	0.0122	0.0144
0.6	0.0024	0.0026	0.0031	0.0037	0.0046	0.0056	0.0068	0.0082	0.0097	0.0115
0.7	0.0018	0.0019	0.0023	0.0028	0.0034	0.0042	0.0051	0.0061	0.0073	0.0086
0.8	0.0012	0.0013	0.0015	0.0018	0.0023	0.0028	0.0034	0.0041	0.0049	0.0057
0.9	0.0006	0.0007	0.0008	0.0009	0.0011	0.0014	0.0017	0.0020	0.0024	0.0029

((12, 33, 11, 4, 3), (12, 26, 6, 2, 3, 6, 20, 1)), a=1.8822, N=737										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0029	0.0032	0.0035	0.0039	0.0045	0.0053	0.0062	0.0072	0.0083	0.0096
0.1	0.0026	0.0028	0.0031	0.0035	0.0041	0.0048	0.0056	0.0065	0.0075	0.0087
0.2	0.0023	0.0025	0.0028	0.0031	0.0037	0.0043	0.0049	0.0058	0.0068	0.0077
0.3	0.0020	0.0022	0.0024	0.0028	0.0032	0.0037	0.0043	0.0050	0.0058	0.0067
0.4	0.0018	0.0019	0.0021	0.0024	0.0027	0.0032	0.0037	0.0043	0.0050	0.0058
0.5	0.0015	0.0016	0.0017	0.0019	0.0023	0.0027	0.0031	0.0036	0.0042	0.0048
0.6	0.0012	0.0013	0.0014	0.0016	0.0018	0.0021	0.0025	0.0029	0.0033	0.0039
0.7	0.0009	0.0009	0.0010	0.0012	0.0014	0.0016	0.0019	0.0022	0.0025	0.0029
0.8	0.0006	0.0006	0.0007	0.0008	0.0009	0.0010	0.0012	0.0014	0.0017	0.0019
0.9	0.0003	0.0003	0.0003	0.0004	0.0005	0.0005	0.0006	0.0007	0.0008	0.0009

((12, 22, 11, 6, 5), (12, 26, 6, 2, 3, 6, 20, 1)), a=2.6815, N=913										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0015	0.0015	0.0016	0.0017	0.0018	0.0021	0.0023	0.0025	0.0028	0.0032
0.1	0.0013	0.0014	0.0014	0.0015	0.0017	0.0019	0.0021	0.0023	0.0025	0.0029
0.2	0.0011	0.0012	0.0013	0.0014	0.0015	0.0016	0.0018	0.0020	0.0023	0.0025
0.3	0.0010	0.0011	0.0011	0.0012	0.0013	0.0014	0.0016	0.0017	0.0019	0.0022
0.4	0.0008	0.0009	0.0009	0.0010	0.0011	0.0012	0.0014	0.0015	0.0017	0.0019
0.5	0.0007	0.0008	0.0008	0.0009	0.0009	0.0010	0.0011	0.0013	0.0014	0.0016
0.6	0.0006	0.0006	0.0006	0.0007	0.0008	0.0008	0.0009	0.0010	0.0011	0.0013
0.7	0.0004	0.0005	0.0005	0.0005	0.0006	0.0006	0.0007	0.0007	0.0009	0.0009
0.8	0.0003	0.0003	0.0003	0.0003	0.0004	0.0004	0.0005	0.0005	0.0007	0.0006
0.9	0.0001	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0003	0.0003	0.0003

((16, 16, 6, 6, 2), (16, 28, 6, 4, 12, 16, 1)), a=1.8394, N=961										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0019	0.0021	0.0023	0.0026	0.0029	0.0034	0.0039	0.0045	0.0052	0.0059
0.1	0.0018	0.0019	0.0021	0.0023	0.0027	0.0031	0.0035	0.0041	0.0047	0.0054
0.2	0.0016	0.0017	0.0018	0.0021	0.0024	0.0027	0.0031	0.0036	0.0042	0.0048
0.3	0.0014	0.0015	0.0016	0.0018	0.0021	0.0024	0.0027	0.0032	0.0036	0.0042
0.4	0.0012	0.0013	0.0014	0.0015	0.0018	0.0020	0.0024	0.0027	0.0031	0.0036
0.5	0.0009	0.0010	0.0011	0.0013	0.0015	0.0017	0.0019	0.0023	0.0026	0.0029
0.6	0.0008	0.0008	0.0009	0.0010	0.0012	0.0014	0.0016	0.0018	0.0021	0.0024
0.7	0.0006	0.0006	0.0007	0.0008	0.0009	0.0010	0.0012	0.0014	0.0016	0.0018
0.8	0.0004	0.0004	0.0005	0.0005	0.0006	0.0007	0.0008	0.0009	0.0010	0.0012
0.9	0.0002	0.0002	0.0002	0.0003	0.0003	0.0003	0.0004	0.0005	0.0005	0.0006

((16, 48, 15, 5, 4), (16, 28, 6, 4, 12, 16, 1)), a=2.6124, N=1217										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0011	0.0012	0.0012	0.0013	0.0014	0.0016	0.0017	0.0019	0.0021	0.0024
0.1	0.0010	0.0010	0.0011	0.0012	0.0013	0.0014	0.0016	0.0017	0.0019	0.0021
0.2	0.0009	0.0009	0.0009	0.0010	0.0011	0.0012	0.0014	0.0015	0.0017	0.0018
0.3	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010	0.0012	0.0013	0.0015	0.0016
0.4	0.0007	0.0007	0.0007	0.0008	0.0008	0.0009	0.0010	0.0011	0.0013	0.0014
0.5	0.0006	0.0006	0.0006	0.0007	0.0007	0.0008	0.0009	0.0009	0.0011	0.0011
0.6	0.0005	0.0005	0.0005	0.0005	0.0006	0.0006	0.0007	0.0008	0.0008	0.0009
0.7	0.0003	0.0003	0.0004	0.0004	0.0004	0.0005	0.0005	0.0006	0.0006	0.0007
0.8	0.0002	0.0002	0.0002	0.0003	0.0003	0.0003	0.0003	0.0004	0.0004	0.0005
0.9	0.0001	0.0001	0.0001	0.0001	0.0001	0.0002	0.0001	0.0002	0.0002	0.0002

((16, 48, 15,5,4), (16, 36,7,4,3,4,32,1)), a=1.5929, N=1345										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0013	0.0015	0.0017	0.0020	0.0024	0.0029	0.0034	0.0041	0.0048	0.0056
0.1	0.0012	0.0013	0.0015	0.0018	0.0022	0.0026	0.0031	0.0037	0.0043	0.0051
0.2	0.0011	0.0011	0.0014	0.0016	0.0019	0.0022	0.0027	0.0033	0.0039	0.0045
0.3	0.0009	0.0010	0.0012	0.0014	0.0017	0.0020	0.0024	0.0028	0.0034	0.0039
0.4	0.0008	0.0009	0.0010	0.0012	0.0014	0.0017	0.0020	0.0024	0.0029	0.0034
0.5	0.0007	0.0007	0.0008	0.0010	0.0012	0.0014	0.0017	0.0020	0.0024	0.0028
0.6	0.0005	0.0006	0.0008	0.0008	0.0009	0.0011	0.0014	0.0016	0.0019	0.0023
0.7	0.0004	0.0004	0.0005	0.0006	0.0007	0.0008	0.0010	0.0012	0.0014	0.0017
0.8	0.0003	0.0003	0.0003	0.0004	0.0005	0.0005	0.0007	0.0008	0.0009	0.0011
0.9	0.0001	0.0001	0.0002	0.0002	0.0002	0.0003	0.0004	0.0004	0.0004	0.0006

References:

- [1] Box, G. E. P. and Hunter, J.S. Multifactor exponential designs for exploring response surfaces, *Annals of Mathematical Statistics*, 1957; 28: 195-241.
- [2] Das M.N. and Narasimham, V.L. (1962), Construction of rotatable designs through balanced incomplete block designs, *Annals of Mathematical Statistics*, 33: 1421-1439.
- [3] Das R. N. Robust second order rotatable designs (part-I). *Calcutta Statistical Association Bulletin*, 1997; 47: 199-214.
- [4] Das R.N. Robust second order rotatable designs (part-II). *Calcutta Statistical Association Bulletin*, 1997; 49: 65-78.
- [5] Das R.N. Robust second order rotatable designs (part-III). *Journal of Indian Society of Agricultural Statistics*, 2003b; 56: 117-130.
- [6] Das, R.N. Slope rotatability with correlated errors, *Calcutta Statistical Association Bulletin*, 2003b; 54: 58-70.
- [7] Hader R.J. and Park S.H. Slope rotatable central composite designs, *Technometrics*, 1978; 20: 413-417.
- [8] Narasimham V. L. Ramachandrarao P. Rao K. N. Construction of second order rotatable designs through a pair of incomplete block designs, *Journal of the Indian Society for Agricultural Statistics*, 1983; 35: 36-4.
- [9] Raghavarao D, (1971) *Constructions and combinatorial problems in design of experiments*, John Wiley, New York.
- [10] Rajyalakshmi K. (2014), Some contributions to second order rotatable and slope rotatable designs under different correlated error structures, unpublished Ph.D. thesis, Acharya Nagarjuna University, Guntur-522510, India.
- [11] Rajyalakshmi K. Victorbabu B. Re. An empirical study on robustness of slope rotatable central composite designs, *Journal of Statistics*, 2014a; 21:1-14.
- [12] Rajyalakshmi K. Victorbabu B. Re. Second order slope rotatable designs under intra-class correlation structure of errors using symmetrical unequal block arrangements with two unequal block arrangements, *Thailand Statistician*, 2014b; 12: 71-82.
- [13] Rajyalakshmi K. Victorbabu B. Re. Second Order slope rotatable designs under intra-class correlated error structure using balanced incomplete block designs, *Thailand Statistician*, 2015; 13: 169-183.

- [14] Sulochana B. Victorbabu B. Re. (2019), A study of second order slope rotatable designs under intra-class correlated structure of errors using a pair of balanced incomplete block designs, *Andhra Agricultural Journal*, 2019; 66:12-20.
- [15] Sulochana B. Victorbabu B. Re. A study of second order slope rotatable designs under intra-class correlated structure of errors using a pair of symmetrical unequal block arrangements with two unequal block sizes. *Journal of Interdisciplinary Cycle Research*, 2020a; 27: 239-247.
- [16] Sulochana B. Victorbabu B. Re. A study of second order slope rotatable designs under intra-class correlated structure of errors using partially balanced incomplete block type designs. *Asian Journal of Probability and Statistics*, 2020b; 7: 15-28.
- [17] Victorbabu B. Re. Construction of second order slope rotatable designs using symmetrical unequal block arrangements with two unequal block sizes. *Journal of the Korean Statistical Society*, 2002; 31, 153-161.
- [18] Victorbabu B. Re. On second order slope rotatable designs- A review. *Journal of the Korean Statistical Society*, 2007; 36:373-386.
- [19] Victorbabu, B. Re. and Narasimham, V. L. Construction of second order slope rotatable designs through balanced incomplete block designs, *Communications in Statistics-Theory Methods*, 1991a; 20: 2467-2478.
- [20] Victorbabu, B. Re, Narasimham, V. L. Construction of second order slope rotatable designs through a pair of incomplete block designs, *Journal Indian society Agricultural Statistics*, 1991b; 43: 291-295.
- [21] Victorbabu B. Re, Ashok D, Narasimham V.L. Construction of second order slope rotatable designs using a pair of dissimilar incomplete block designs. *ProbStat Forum*, 2009; 2:1-7.