Picard and Adomian solutions of a nonlocal Cauchy problem of a delay differential equation

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Abstract

In this paper, two methods are used to solve a nonlocal Cauchy problem of a delay differential equation; Adomian decomposition method (ADM) and Picard method. The existence and uniqueness of the solution are proved. The convergence of the series solution and the error analysis are studied.
Keywords: Nonlocal Cauchy problem, existence, uniqueness, error analysis, Adomian method, Picard method.
MSC: [2000]34A12, 34A30, 34D20

1. Introduction

In this paper we concerned with the analytical solution of a nonlocal Cauchy problem of a delay differential equation which have many applications in engineering and science, including electrical networks, control theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology ([1]-[16]). We use Adomain decomposition method ([17]-[24]) for solving this type of equations. The existence and uniqueness of the solution will prove. The convergence of ADM series solution will discuss and the error analysis is given. This method has many advantages; it is efficiently works with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization. We compere ADM solution with Picard solution in the given numerical examples.

Here we are concerned with the nonlocal Cauchy problem of the delay differential equation

$$\frac{dx(t)}{dt} = f\big(t, x(t-r)\big), \quad t \in (0,T], \quad r > 0 \tag{1}$$

$$x(t) = x_0, \quad t < 0, \tag{2}$$

with the nonlocal condition

$$x(0) = \sum_{k=1}^{n} a_k\, x(t_k), \quad t_k \in (r,T). \tag{3}$$

The existence and uniqueness of the solution $x \in C(J)$, where $C(J)$ is the space of all continuous functions and $J = [0,T]$, $T < \infty$ of the nonlocal problem (1)-(3) will be proved, the integral representation of this solution will be proved, the solution algorithm using ADM will be given and the converge of the series solution is proved.

2. Problem Solving

2.1. Integral representation

For the integral representation of the solution of the nonlocal problem (1)-(3) we have the following lemma.

Lemma 1 If $(1-\sum_{k=1}^{n} a_k) > 0$, then the nonlocal problem (1)-(3) and the integral equation

$$x(t) = \left(1-\sum_{k=1}^{n} a_k\right)^{-1}\left(\sum_{k=1}^{n} a_k \int_0^r f(s,x_0)ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f\big(s,x(s-r)\big)ds\right) + \int_0^r f(s,x_0)ds + \int_r^t f\big(s,x(s-r)\big)ds. \quad (4)$$

are equivalent.

Proof. Operating with $I = \int_0^t (.)\, ds$ to both sides of equation (1), we get

$$x(t) = x(0) + \int_0^r f(s,x_0)ds + \int_r^t f\big(s,x(s-r)\big)ds. \quad (5)$$

Let $t = t_k$ in equation (5), then we get

$$x(t_k) = x(0) + \int_0^r f(s,x_0)ds + \int_r^{t_k} f\big(s,x(s-r)\big)ds, \quad (6)$$

$$\sum_{k=1}^{n} a_k\, x(t_k) = x(0)\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k \int_0^r f(s,x_0)ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f\big(s,x(s-r)\big)ds. \quad (7)$$

Substitute from equation (3) into equation (7) we get,

$$x(0) = x(0)\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k \int_0^r f(s,x_0)ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f\big(s,x(s-r)\big)ds, \quad (8)$$

$$x(0) - x(0)\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_k \int_0^r f(s,x_0)ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f\big(s,x(s-r)\big)ds \quad (9)$$

And

$$x(0) = \left(1-\sum_{k=1}^{n} a_k\right)^{-1}\left(\sum_{k=1}^{n} a_k \int_0^r f(s,x_0)ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f\big(s,x(s-r)\big)ds\right) \quad (10)$$

Substitute from equation (8) into equation (5) we obtain (4).
To complete the proof, differentiating (4) we obtain (1). Also, let $t = 0$ in (4) and then by direct calculations, we can get (3).

2.2. The solution algorithm

The solution algorithm of equation (4) using ADM is

$$x_0(t) = \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_0^r f\,(s, x_0) ds\right) + \int_0^r f\,(s, x_0) ds\,, \tag{11}$$

$$x_m(t) = \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} A_{m-1}\,(s-r) ds\right) + \int_r^{t} A_{m-1}\,(s-r) ds. \tag{12}$$

Where A_m are Adomian polynomials of the nonlinear term $f\big(t, x(t-r)\big)$ that take the following form

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[f\left(t, \sum_{i=0}^{\infty} \lambda^i x_i(t-r)\right)\right]_{\lambda=0} \tag{13}$$

Finally, the solution of problem (1)-(3) will be

$$x(t) = \sum_{i=0}^{\infty} x_i\,(t). \tag{14}$$

3. Convergence Analysis

3.1. Existence and Uniqueness theorem

Define the mapping $F: E \to E$ where E is the Banach space $(C(J), \|\cdot\|)$ of all continuous functions on J with the norm $\|x\| = \max_{t\in J} |x(t)|$.

Assume now that the function $f: [0, T] \times R \to R$ is continuous and satisfies the Lipschitz condition

$$|f(t, x(t-r)) - f(t, y(t-r))| \le k|x(t-r) - y(t-r)| \tag{15}$$

Theorem 1: *Let f satisfies the Lipschitz condition (15), then the integral equation (4) which equivalent to problem (1)-(3), has a unique solution $x \in C(J)$.*

Proof: The mapping $F: E \to E$ defined as,

$$Fx = \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_0^r f\,(s, x_0) ds + \sum_{k=1}^{n} a_k \int_r^{t_k} f\,\big(s, x(s-r)\big) ds\right)$$

Let $x, y \in E$, then

$$Fx - Fy = \left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} \left[f\big(s, x(s-r)\big) - f\big(s, y(s-r)\big)\right] ds\right)$$

Which implies that

$$|Fx - Fy| = \left|\left(1-\sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} \left[f\big(s, x(s-r)\big) - f\big(s, y(s-r)\big)\right] ds\right)\right.$$

$$\max_{t\in J}|Fx-Fy| \le k\left[\left(1-\sum_{k=1}^{n}a_k\right)^{-1}\sum_{k=1}^{n}a_k\ \max_{t\in J}\int_{r}^{t_k}|x(s-r)-y(s-r)|\,ds\right.$$

$$\|Fx-Fy\| \le k\left[\left(1-\sum_{k=1}^{n}a_k\right)^{-1}\sum_{k=1}^{n}a_k\int_{r}^{t_k}d\,s+\int_{r}^{t}d\,s\right]\|x-y\|$$

Now, if $k(T-r)[(1-\sum_{k=1}^{n}a_k)^{-1}(\sum_{k=1}^{n}a_k\ )+1]<1$, then we get

$$\|Fx-Fy\| \le \|x-y\|,$$

Therefore the mapping F is contraction and there exists a unique solution $x\in C(J)$ to the nonlocal Cauchy problem (1)-(3) given by (4), where

$$x(0)=\lim_{t\to 0}x\,(t)\ =\left(1-\sum_{k=1}^{n}a_k\right)^{-1}\left(\sum_{k=1}^{n}a_k\int_{0}^{r}f\,(s,x_0)ds\right.$$
$$\left.+\sum_{k=1}^{n}a_k\int_{r}^{t_k}f\left(s,x(s-r)\right)ds\right)$$

And

$$x(T)=\lim_{t\to T}\ x(t)\ =\left(1-\sum_{k=1}^{n}a_k\right)^{-1}\left(\sum_{k=1}^{n}a_k\int_{0}^{r}f\,(s,x_0)ds\right.$$
$$\left.+\sum_{k=1}^{n}a_k\int_{r}^{t_k}f\left(s,x(s-r)\right)ds\right)$$

This completes the proof. ■

3.2. Proof of convergence

<u>Theorem 2</u>: *The series solution (14) of the problem (1)-(3) using ADM converges if* $|x_1(t)|<c, c$ *is a positive constant.*

<u>Proof:</u> Define the sequence $\{S_p\}$ such that, $S_p=\sum_{i=0}^{p}x_i\,(t)$ is the sequence of partial sums from the series solution $\sum_{i=0}^{\infty}x_i\,(t)$ since,

$$f\left(t,x(t-r)\right)=\sum_{i=0}^{\infty}A_i,$$

So,

$$f\left(t,S_p\right)=\sum_{i=0}^{p}A_i,$$

From equations (12) and (13) we have,

$$\sum_{i=0}^{\infty}x_i=\left(1-\sum_{k=1}^{n}a_k\right)^{-1}\left(\sum_{k=1}^{n}a_k\int_{0}^{r}f\,(s,x_0)ds+\sum_{k=1}^{n}a_k\int_{r}^{t_k}\sum_{i=0}^{\infty}A_{i-1}\,(s-r)ds\right)$$

Let S_p and S_q be two arbitrary partial sums with $p>q$, then we get,

$$S_p = \sum_{i=0}^{p} x_i = \left(1 - \sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_0^r f\,(s, x_0) ds + \sum_{k=1}^{n} a_k \int_r^{t_k} \sum_{i=1}^{p} A_{i-1}\,(s-r) ds \right)$$

And

$$S_q = \sum_{i=0}^{q} x_i = \left(1 - \sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_0^r f\,(s, x_0) ds + \sum_{k=1}^{n} a_k \int_r^{t_k} \sum_{i=1}^{q} A_{i-1}\,(s-r) ds \right)$$

Now, we are going to prove that $\{S_p\}$ is a Cauchy sequence in this Banach space$E$.

$$S_p - S_q = \left(1 - \sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} \left[\sum_{i=1}^{p} A_{i-1}\,(s) - \sum_{i=1}^{q} A_{i-1}\,(s) \right] ds \right)$$

$$= \left(1 - \sum_{k=1}^{n} a_k\right)^{-1} \left(\sum_{k=1}^{n} a_k \int_r^{t_k} \left[\sum_{i=q+1}^{p} A_{i-1} \right] ds \right) + \int_r^{t} \left[\sum_{i=q+1}^{p} A_{i-1} \right] ds$$

Let $p = q + 1$ then,

$$\|S_{q+1} - S_q\| \leq \beta \|S_q - S_{q-1}\| \leq \beta^2 \|S_{q-1} - S_{q-2}\| \leq \cdots \leq \beta^q \|S_1 - S_0\|$$

From the triangle inequality we have,

$$\|S_p - S_q\| \leq \|S_{q+1} - S_q\| + \|S_{q+2} - S_{q+1}\| + \cdots + \|S_p - S_{p-1}\|$$

Since, $0 < \beta = k(T - r)[(1 - \sum_{k=1}^{n} a_k)^{-1} (\sum_{k=1}^{n} a_k) + 1] < 1$ and $p > q$ then $(1 - \beta^{p-q}) \leq 1$. Consequently,

$$\|S_p - S_q\| \leq \frac{\beta^q}{1 - \beta} \|x_1\|$$

However, $|x_1(t)| < c$ and as $q \to \infty$ then, $\|S_p - S_q\| \to 0$ and hence, $\{S_p\}$ is a Cauchy sequence in this Banach space E so, the series $\sum_{i=0}^{\infty} x_i\,(t)$ converges. ■

3.3. Error analysis

<u>Theorem 3</u>: *The maximum absolute truncation error of the solution (14) to the problem (1)-(3) estimated to be,*

$$\left\| x - \sum_{i=0}^{q} x_i \right\| \leq \frac{\beta^q}{1 - \beta} \|x_1\|$$

<u>Proof:</u> From Theorem 2 we have,

$$\|S_p - S_q\| \leq \frac{\beta^q}{1 - \beta} \max_{t \in J} |x_1(t)|$$

But $S_p = \sum_{i=0}^{p} x_i\,(t)$ as $p \to \infty$ then $S_p \to x(t)$ so,

$$\|x - S_q\| \leq \frac{\beta^q}{1 - \beta} \|x_1\|$$

Therefore, the maximum absolute truncation error in the interval J is,

$$\left\| x - \sum_{i=0}^{q} x_i \right\| \leq \frac{\beta^q}{1 - \beta} \|x_1\|$$

4. Numerical Examples

The following examples will solve by using ADM method and the solution will compare by using Picard method.

Example 1 Let $\alpha > 0$, consider the following example,

$$\frac{dx}{dt} = \frac{1}{20}x^2(t-0.1), \qquad t \in (0,10], \tag{16}$$

$$x(t) = 1, \quad t < 0, \tag{17}$$

$$x(0) = \alpha x\left(\frac{1}{2}\right). \tag{18}$$

We prove here, firstly, that as $\alpha \to 0$ the solution of this nonlocal problem continuo to the solution of the usual Cauchy problem (with $\alpha = 0$). This proves the validity of our algorithm. Using equation (10), problem (16)-(18) will be

$$x(t) = \frac{\alpha}{1-\alpha}\left[\int_0^{0.1} \frac{1}{20} ds + \int_{0.1}^{1/2} \frac{1}{20}x^2(s-0.1)ds\right] + \int_0^{0.1} \frac{1}{20} ds + \int_{0.1}^{t} \frac{1}{20}x^2(s-0.1)ds \tag{19}$$

Solution using ADM method

Applying ADM to equation (19), we have

$$x_0(t) = \frac{0.005}{1-\alpha}, \tag{20}$$

$$x_i(t) = \frac{\alpha}{20(1-\alpha)}\int_{0.1}^{1/2} A_{i-1}(s-0.1)ds + \frac{1}{20}\int_{0.1}^{t} A_{i-1}(s-0.1)ds, \; i \geq 1. \tag{21}$$

From equations (20) and (21), the solution of the problem (16)-(18) is,

$$x(t) = \sum_{i=0}^{m} x_i(t). \tag{22}$$

Solution using Picard method

Applying Picard method to equation (19), we have

$$x_0(t) = \frac{0.005}{1-\alpha}, \tag{23}$$

$$x_i(t) = \frac{0.005}{1-\alpha} + \frac{\alpha}{20(1-\alpha)}\int_{0.1}^{1/2} x_{i-1}^2(s-0.1)ds + \frac{1}{20}\int_{0.1}^{t} x_{i-1}^2(s-0.1)ds, \; i \geq 1. \tag{24}$$

The solution of the problem (16)-(18) using Picard method will be,

$$x(t) = x_m(t). \tag{25}$$

Figures 1.a - 1.d show a comparison between ADM and Picard solutions (when $\alpha = 0.1, 0.001, 0.00001, 0$ respectively, and $m = 5$).

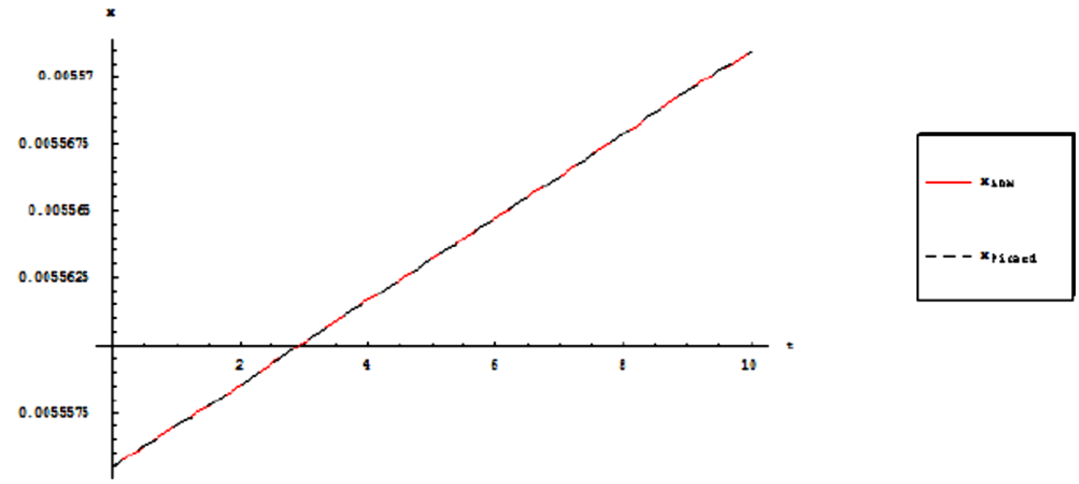

Fig (1-a): ADM and Picard solutions [$\alpha = 0.1$]

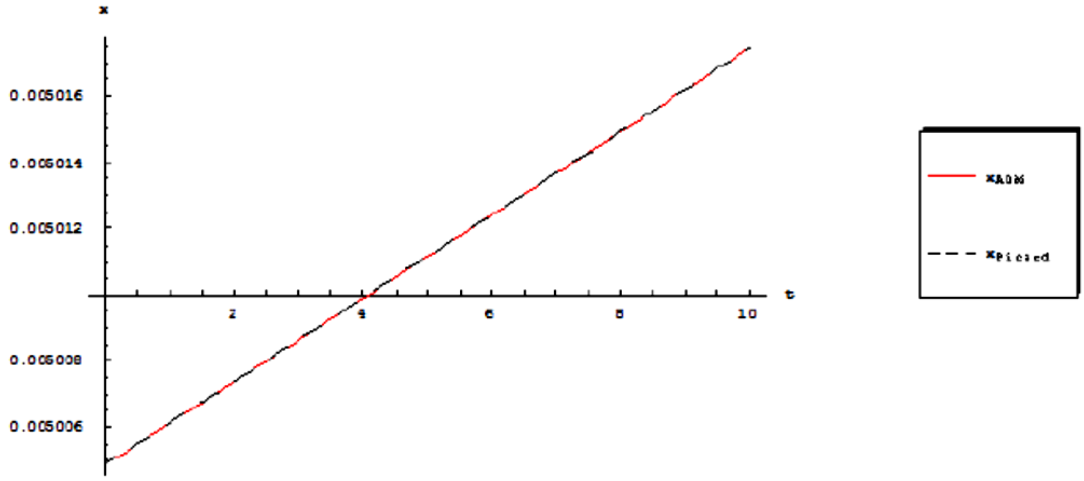

Fig (1-b): ADM and Picard solutions [$\alpha = 0.001$]

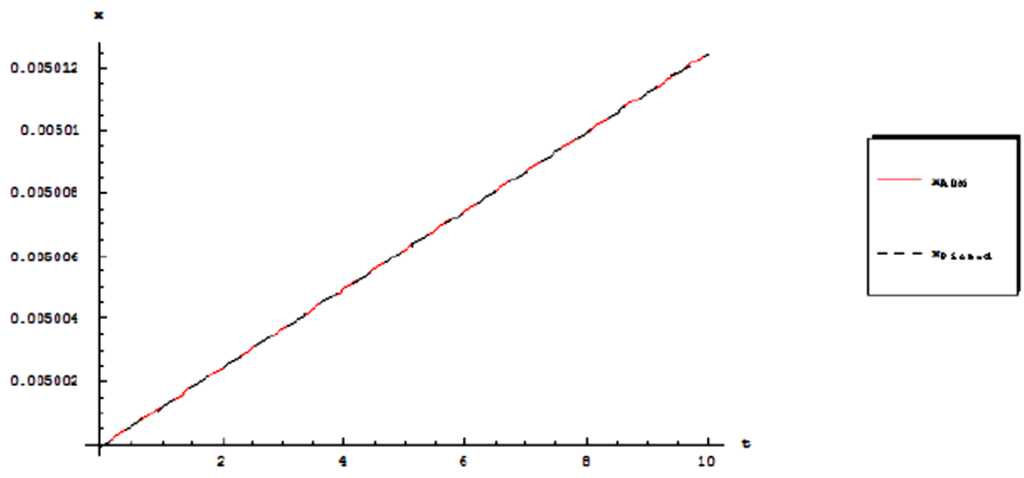

Fig (1-c): ADM and Picard solutions $[\alpha = 0.00001]$

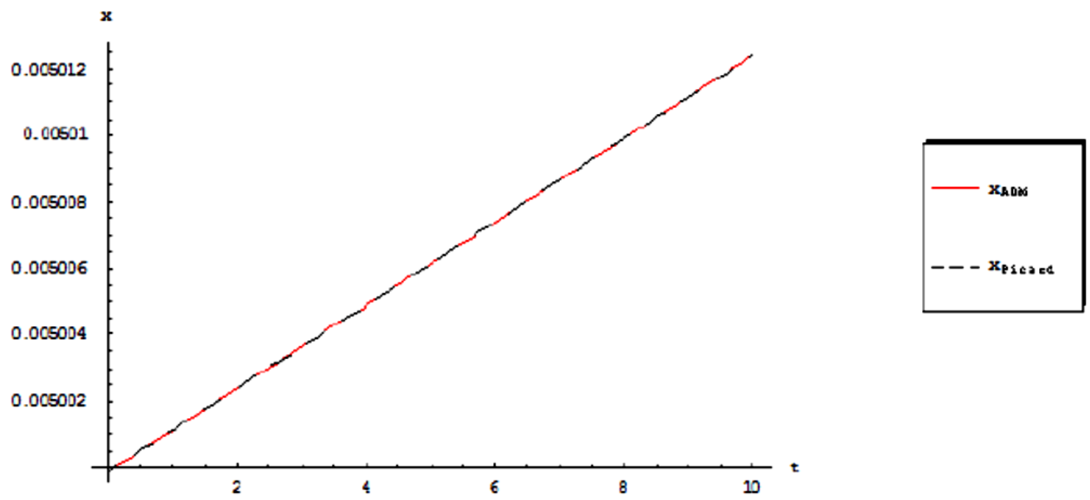

Fig (1-d): ADM and Picard solutions $[\alpha = 0]$

Table (1. a) shows the absolute error between ADM solution and Picard solution when (m=5, $\alpha = 0.1$).

Table (1. a)

t	***Absolute error***
1	$\mathbf{4.2304 \times 10^{-19}}$
2	$\mathbf{4.2369 \times 10^{-19}}$
3	$\mathbf{4.25117 \times 10^{-19}}$
4	$\mathbf{4.30775 \times 10^{-19}}$
5	$\mathbf{4.50657 \times 10^{-19}}$
6	$\mathbf{5.06871 \times 10^{-19}}$
7	$\mathbf{6.40971 \times 10^{-19}}$
8	$\mathbf{9.23075 \times 10^{-19}}$
9	$\mathbf{1.46274 \times 10^{-18}}$

10	$\mathbf{2.42164 \times 10^{-18}}$

Table (1.b) shows a comparison between the time of ADM solution and Picard solution (*m=5*, $\alpha = 0.1$).

Table (1.b)

ADM time	*Picard time*
0.155 sec.	**0.234 sec.**

Example 2 *Consider the following nonlocal DE,*

$$\frac{dx}{dt} = \frac{1}{10} t^2 e^{x^2(t-0.5)}, \qquad t \in (0,4], \tag{26}$$

$$x(t) = \frac{1}{2}, \quad t < 0, \tag{27}$$

$$x(0) = \frac{1}{2} x(0.7) - \frac{1}{4} x(0.9). \tag{28}$$

Using equation (10), problem (26)-(28) will be

$$x(t) = \frac{e^{1/4}}{30} \int_0^{0.5} s^2\, ds + \frac{1}{15} \int_{0.5}^{0.7} \left[s^2 e^{x^2(s-0.5)}\right] ds - \frac{1}{30} \int_{0.5}^{0.9} \left[s^2 e^{x^2(s-0.5)}\right] ds$$

$$+ \frac{e^{1/4}}{10} \int_0^{0.5} s^2\, ds + \frac{1}{10} \int_{0.5}^{t} s^2\, e^{x^2(s-0.5)} ds, \tag{29}$$

<u>Solution using ADM method</u>

Applying ADM to equation (29), we have

$$x_0(t) = \frac{e^{1/4}}{30} \int_0^{0.5} s^2\, ds + \frac{e^{1/4}}{10} \int_0^{0.5} s^2\, ds, \tag{30}$$

$$x_i(t) = \frac{1}{15} \int_{0.5}^{0.7} s^2\, A_{i-1}(s-0.5) ds - \frac{1}{30} \int_{0.5}^{0.9} s^2\, A_{i-1}(s-0.5) ds$$

$$+ \frac{1}{10} \int_{0.5}^{t} s^2\, A_{i-1}(s-0.5) ds, \qquad i \geq 1. \tag{31}$$

From equations (30) and (31), the solution of the problem (26)-(28) is,

$$x(t) = \sum_{i=0}^{m} x_i\,(t). \tag{32}$$

<u>Solution using Picard method</u>

Applying Picard method to equation (29), we have

$$x_0(t) = \frac{e^{1/4}}{30} \int_0^{0.5} s^2\, ds + \frac{e^{1/4}}{10} \int_0^{0.5} s^2\, ds, \tag{33}$$

$$x_i(t) = \frac{e^{1/4}}{30} \int_0^{0.5} s^2\, ds + \frac{e^{1/4}}{10} \int_0^{0.5} s^2\, ds + \frac{1}{15} \int_{0.5}^{0.7} s^2\, e^{x_{i-1}^2(s-0.5)} ds$$

$$-\frac{1}{30}\int_{0.5}^{0.9} s^2\, e^{x_{i-1}^2(s-0.5)}ds \;+\frac{1}{10}\int_{0.5}^{t} s^2\, e^{x_{i-1}^2(s-0.5)}ds, \quad i \geq 1. \qquad (34)$$

The solution of the problem (26)-(28) using Picard method will be,

$$x(t) = x_m(t). \qquad (35)$$

Figure 2 shows a comparison between ADM solution (when$m = 5$) and Picard solution (when$m = 2$).

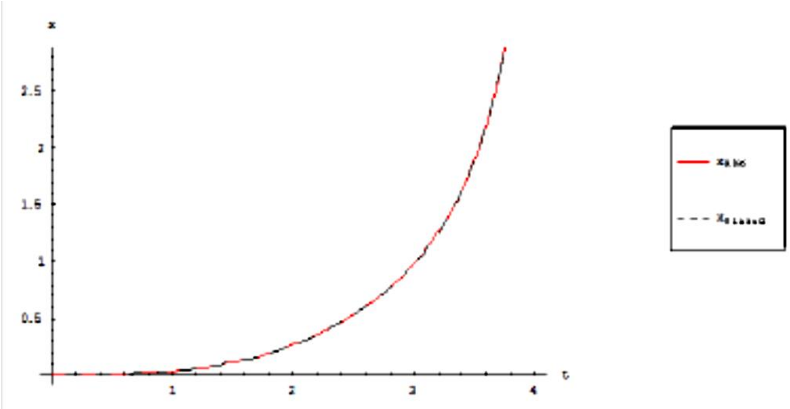

Fig (2): ADM and Picard solutions.

Table (2. a) shows the absolute error between ADM solution (when $m=5$) and Picard solution (when $m=2$).

Table (2. a)

t	*Absolute error*
0.5	**2.57408×10^{-12}**
1	**2.43143×10^{-11}**
1.5	**2.11219×10^{-9}**
2	**1.65101×10^{-7}**
2.5	**0.000032022**
3	**0.0011126**
3.5	**0.0105128**
4	**0.262885**

Table (2. b) shows a comparasion between the time of ADM solution(when $m=5$) and Picard solution(when $m=2$).

Table (2.b)

ADM time	*Picard time*
0.296 sec.	5.693 sec.

5. Conclusion

In this paper, we use two interesting methods (ADM and Picard) to solve a nonlocal Cauchy problem of a delay differential equation. These two methods give analytical solution, when we comparing the results of the two methods, we see that the two methods give very enclosed solutions but when we compere the taken time of solution of the two methods, we see that ADM take time lese than Picard method.

6. Acknowledgments

I wish to thank Professor Ahmed M. A. El-Sayed (Mathematics Department, Faculty of Science, Alexandria University, Alexandria, Egypt) for his help, support and encouragement of this study.

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