Journal of Advances in Mathematics and Computer Science
X(X): XX-XX, 20XX

SCIENCEDOMAIN *international*
www.sciencedomain.org

Exact steady solutions for the two dimensional Broadwell model

Amah Séna d'Almeida[*1] and Kokou Anani Agosseme[2]

[1] *Université de Lomé (Togo) - Département de mathématiques, dal_me@yahoo.fr*
[2] *Université de Lomé (Togo) - Département de mathématiques, ka.agosseme@gmail.com*

Short Research Articles

Received: 18 November 2019
Accepted: XX XX 20XX
Online Ready: XX XX 20XX

Abstract

Existence and boundedness of the solutions of the boundary value problem for the four velocity two dimensional Broadwell model for bounded boundary conditions is proved and exact analytic solutions are built. An application to the determination of the accommodation coefficients on the boundaries of a flow in a box is performed.

Keywords: Discrete models; kinetic theory; Boltzmann equation; Rarefied gas; Steady flow.

1 Introduction

A discrete velocity model replaces the nonlinear integro-differential Boltzmann equation by a set of semi-linear hyperbolic partial differential equations which leads to quantitative and qualitative interesting results in the study of several problems of gas dynamics [Broadwell (1964); d'Almeida (2008); d'Almeida and Gatignol (2003); Platkowski and Illner (1988)]. An advantage of discrete kinetic theory is the possibility to find exact analytic solutions. Various exact solutions have been built for discrete velocity models in the one dimensional case [Natta et al. (2018); d'Almeida and Gatignol (1995); d'Almeida (2007); Cornille and d'Almeida (2002)]. The situation is quite different for two dimensional problems even in the steady case. In the pioneering work [Cercignani et al. (1988)] the problem of existence of a solution for the two dimensional four velocity Broadwell model is investigated and the existence of solution proved. In this work, we prove the existence and the boundedness of the solutions of the boundary value problem and build exact analytic solutions. The solutions are not unique in general. The paper is organized as follows. In the section 2 we briefly describe the model, state the boundary value problem and present the main result of the paper which is proved in section 3. The exact analytic solution are presented in the section 4 and an application to the determination of accommodation coefficients is performed for a gas flow in a box in section 5.

**Corresponding author: E-mail: dal_me@yahoo.fr*

2 Statement of the problem

The steady flow of a gas in a rectangular box is a problem of gas dynamics the modelling of which can lead to the boundary value problem.
We choose the origin O of the orthonormal reference $(O, \vec{e}_1, \vec{e}_2)$ of $\mathbb{R}^2$ so that the edges of the box are located on the lines $x = 0$, $x = a$, $y = 0$ and $y = b$, $0 < b \leq a$. The velocities of the Broadwell model in the basis $(\vec{e}_1, \vec{e}_2)$ are: $\vec{u}_1 = c(1,0)$, $\vec{u}_3 = c(0,1)$, $\vec{u}_{i+1} = -\vec{u}_i$, $i = 1, 3$ where c is an arbitrary positive constant. We denote by $N_i(t', x', y')$ the number density of particles of velocity $\vec{u}_i$ in point $M(x', y')$ at time t'. The N_i are continuous functions of t', x' and y'. The kinetic equations for this model [Cabannes (1980)] are:

$$\begin{cases} \dfrac{\partial N_1}{\partial t'} + c\dfrac{\partial N_1}{\partial x'} & = & cs\left(N_3N_4 - N_1N_2\right) = Q'(N) \\ \dfrac{\partial N_2}{\partial t'} - c\dfrac{\partial N_2}{\partial x'} & = & Q'(N) \\ \dfrac{\partial N_3}{\partial t'} + c\dfrac{\partial N_3}{\partial y'} & = & -Q'(N) \\ \dfrac{\partial N_4}{\partial t'} - c\dfrac{\partial N_4}{\partial y'} & = & -Q'(N) \end{cases} \tag{2.1}$$

where s is the gas particles collision cross section.

For a steady flow, the boundary value problem has the form:

$$\begin{cases} \dfrac{\partial N_1}{\partial x'} & = & s\left(N_3N_4 - N_1N_2\right) = Q'(N) \\ \dfrac{\partial N_2}{\partial x'} & = & -Q'(N) \\ \dfrac{\partial N_3}{\partial y'} & = & -Q'(N) \\ \dfrac{\partial N_4}{\partial y'} & = & Q'(N) \\ N_1(0, y') & = & \phi'_1(y') \\ N_2(a, y') & = & \phi'_2(y') \\ N_3(x', 0) & = & \phi'_3(x') \\ N_4(x', b) & = & \phi'_4(x') \end{cases} \tag{2.2}$$

The functions ϕ'_k, $k = 1, 2, 3, 4$ are non negative.

The main result of the paper is:

Theorem 2.1. *The problem (2.2) has bounded solution* $N = (N_1, N_2, N_3, N_4)$ *for bounded boundary data* ϕ'_k, $k = 1, 2, 3, 4$.

3 Existence and boundedness of the solution

Let $N = (N_1, N_2, N_3, N_4)$ the total macroscopic density, $\rho^+(N) = N_1 + N_2$ and $\rho^-(N) = N_3 + N_4$ the partial densities of particles whose velocities have non zero component respectively on the $(O, \vec{e}_1)$

and $(O,\vec{e}_2)$ axes. We consider for $\sigma > 0$ the following problem:

$$\left\{\begin{array}{lllll}\dfrac{\partial N_1}{\partial x'}+\sigma N_1\rho^+(N) & = & Q(N)+\sigma N_1\rho^+(N) & = & Q_1^\sigma(N)\\ \dfrac{\partial N_2}{\partial x'}+\sigma N_2\rho^+(N) & = & -Q(N)+\sigma N_2\rho^+(N) & = & Q_2^\sigma(N)\\ \dfrac{\partial N_3}{\partial y'}+\sigma N_3\rho^-(N) & = & -Q(N)+\sigma N_3\rho^-(N) & = & Q_3^\sigma(N)\\ \dfrac{\partial N_4}{\partial y'}+\sigma N_4\rho^-(N) & = & Q(N)+\sigma N_4\rho^-(N) & = & Q_4^\sigma(N)\\ N_1(0,y') & = & \phi_1'(y') & & \\ N_2(a,y') & = & \phi_2'(y') & & \\ N_3(x',0) & = & \phi_3'(x') & & \\ N_4(x',b) & = & \phi_4'(x') & & \end{array}\right. \tag{3.1}$$

Proposition 3.1. *The problem (3.1)is equivalent to problem (2.2).*

Proof. The system (3.1) is obtained from system (2.2) by adding $\sigma N_i\rho^{\pm}(N)$ to the two members of the kinetic equation for N_i so the proof is obvious. □

3.1 Existence of solutions of (3.1)

Let $J = [0,a]\times[0,b]$. We denote by $\mathscr{C}$ the set of continuous functions defined on J, and by $\mathscr{C}_+$ its subset of non negative functions. $\mathscr{C}^4$ and $\mathscr{C}_+^4$ respectively denote their cartesian products.
We introduce the following norms :
If $z=(x',y')\in J$ and $M=(M_1,\ldots,\ M_4)\in\mathscr{C}^4$ then $\|z\|=|x'|+|y'|$, $\|M_i\|=\sup_{\|z\|\le a+b}|M_i(z)|$ and $\|M\|=\sup_{i\in\Lambda}\|M_i\|$, with $\Lambda=\{1,2,3,4\}$.

Theorem 3.1. *The problem* (3.1) *has a solution which belongs to* $\mathscr{C}_+^4$ *for sufficiently large* σ.

For the proof, consider for $M\in\mathscr{C}_+^4$, the following boundary value problem:

$$\left\{\begin{array}{lll}\dfrac{\partial N_1}{\partial x'}+\sigma N_1\rho^+(M) & = & Q(M)+\sigma M_1\rho^+(M)=Q_1^\sigma(M)\\ \dfrac{\partial N_2}{\partial x'}+\sigma N_2\rho^+(M) & = & -Q(M)+\sigma M_2\rho^+(M)=Q_2^\sigma(M)\\ \dfrac{\partial N_3}{\partial y'}+\sigma N_3\rho^-(M) & = & -Q(M)+\sigma M_3\rho^-(M)=Q_3^\sigma(M)\\ \dfrac{\partial N_4}{\partial y'}+\sigma N_4\rho^-(M) & = & Q(M)+\sigma M_4\rho^-(M)=Q_4^\sigma(M)\\ N_1(0,y') & = & \phi_1'(y')\\ N_2(a,y') & = & \phi_2'(y')\\ N_3(x',0) & = & \phi_3'(x')\\ N_4(x',b) & = & \phi_4'(x')\end{array}\right. \tag{3.2}$$

Lemma 3.2. *The problem* (3.2) *has for given* $M\in\mathscr{C}_+^4$ *an unique solution which belongs to* $\mathscr{C}_+^4$ *for sufficiently large* σ..

Proof. The problem (3.2) is a linear problem associated with (3.1) and it is solved by splitting it into the two following boundary value problems :

$$\left\{\begin{array}{lll}\dfrac{\partial N_1}{\partial x'}+\sigma N_1\rho^+(M) & = & Q_1^\sigma(M)\\ \dfrac{\partial N_2}{\partial x'}+\sigma N_2\rho^+(M) & = & Q_2^\sigma(M)\\ N_1(0,y') & = & \phi_1'(y')\\ \widetilde{N}_2(a,y') & = & \phi_2'(y')\end{array}\right. \tag{3.3}$$

and

$$\left\{\begin{array}{lcl} \dfrac{\partial N_3}{\partial y'} + \sigma N_3 \rho^-(M) & = & Q_3^\sigma(M) \\ \dfrac{\partial N_4}{\partial y'} + \sigma N_4 \rho^-(M) & = & Q_4^\sigma(M) \\ N_3(x',0) & = & \phi_3'(x') \\ N_4(x',b) & = & \phi_4'(x') \end{array}\right. \tag{3.4}$$

the unique solution of (3.2) is given by :

$$\begin{array}{lcl} N_1(x',y') & = & \phi_1'(y')\, g^+(x,y) + \displaystyle\int_0^{x'} Q_1^\sigma(M)(s,y') f^+(x'-s,y')ds, \\ N_2(x',y') & = & \phi_2'(y')\, f^+(x'-a,y') - \displaystyle\int_{x'}^{a} Q_2^\sigma(M)(s,y') f^+(x'-s,y')ds, \\ N_3(x',y') & = & \phi_3'(x')\, g^-(x',y') + \displaystyle\int_0^{y'} Q_3^\sigma(M)(s,y') f^-(x',y'-s)ds, \\ N_4(x',y') & = & \phi_4'(x')\, f^-(x',y'-b) - \displaystyle\int_{y}^{b} \widetilde{Q}_4^\sigma(M)(x',s) f^-(x',y'-s)ds. \end{array} \tag{3.5}$$

with:

$$\begin{array}{l} g^+(x',y') = \exp\left(-\sigma \int_0^{x'} \rho^+(M)(a,y')da\right), \quad g^-(x',y') = \exp\left(-\sigma \int_0^{y'} \rho^-(M)(x',a)da\right) \\ f^+(x'-a,y') = \dfrac{g^+(x',y')}{g^+(a,y')}, \quad f^-(x',y'-a) = \dfrac{g^-(x',y')}{g^-(x',a)} \end{array} \tag{3.6}$$

For sufficiently large σ, Q_i^σ is positive $\forall i \in \Lambda$. Hence as ϕ_i' is positive $\forall i \in \Lambda$, $N_i(x',y') > 0$, $i = 1,3$, $\forall (x',y') \in J$ and $N_i(x',y') > 0$, $i = 2,4$ $\forall (x',y') \in J$ if and only if:

$$\begin{array}{l} \phi_2'(y') > \dfrac{\int_{x'}^{a} Q_2^\sigma(M)(s,y') f^+(x'-s,y')ds}{f^+(x'-a,y')} = \displaystyle\int_{x'}^{a} Q_2^\sigma(M)(s,y') f^+(a-s,y')ds, \\ \phi_4'(x') > \dfrac{\int_{y'}^{b} Q_4^\sigma(M)(x',s) f^-(x',y'-s)ds}{f^-(x',y'-b)} = \displaystyle\int_{y'}^{b} Q_4^\sigma(M)(x',s) f^-(x',b-s)ds. \end{array} \tag{3.7}$$

As $0 < f^+(a-s,y') < 1$, $\forall (s,y') \in J$ and $0 < f^-(x',b-s) < 1$, $\forall (x',s) \in J$ we have

$$\begin{array}{l} \int_{x'}^{a} Q_2^\sigma(M)(s,y') f^+(a-s,y')ds \leq a \sup_{(x',y')\in J} Q_2^\sigma(M) \\ \int_{y'}^{b} Q_4^\sigma(M)(x',s) f^-(x',b-s)ds \leq b \sup_{(x',y')\in J} Q_4^\sigma(M) \end{array} \tag{3.8}$$

and it sufficient that $\phi_2' > a \sup_{(x',y')\in J} Q_2^\sigma(M)$ and $\phi_4' > b \sup_{(x',y')\in J} Q_4^\sigma(M)$ to have $N \in \mathscr{C}_+^4$. □

Thus for $\sigma > 0$ the operator $\mathcal{T}_\sigma$ defined by $\mathcal{T}_\sigma(M) = N$ where N is the unique solution of (3.2) is well defined and satisfies:

Lemma 3.3. *$\mathcal{T}_\sigma$ is continuous and compact on J.*

Proof. We have $\mathcal{T}_\sigma(M) = N$ if and only if N is given by the relations (3.6) from which we deduce:

$$\begin{array}{lcl} |N_1(x',y')| & \leqslant & |\phi_1'(y')| \left|g^+(x',y')\right| + \left|\int_0^{x'} Q_1^\sigma(M)(s,y') f^+(x'-s,y')ds\right|, \\ |N_2(x',y')| & \leqslant & |\phi_2'(y')| \left|f^+(x'-a,y')\right| + \left|\int_a^{x'} Q_2^\sigma(M)(s,y') f^+(x'-s,y')ds\right|, \\ |N_3(x',y')| & \leqslant & |\phi_3'(x')| \left|g^-(x',y')\right| + \left|\int_0^{y'} Q_3^\sigma(M)(x',s) f^-(x',y'-s)ds\right|, \\ |N_4(x',y')| & \leqslant & |\phi_4'(x')| \left|f^-(x',y'-b)\right| + \left|\int_b^{y'} Q_4^\sigma(M)(x',s) f^-(x',y'-s)ds\right|. \end{array} \tag{3.9}$$

In one hand using the Generalized Mean Value Theorem, as g^+ and g^- are strictly positive functions, we can find $c_1 \in]0, x'[$, $c_2 \in]x', a[$, $c_3 \in]0, y'[$ and $c_4 \in]y', b[$ such that

$$\begin{array}{rcl} \int_0^{x'} Q_1^\sigma(M)(s,y')f^+(x'-s,y')ds &=& Q_1^\sigma(M)(c_1,y')\int_0^{x'} f^+(x'-s,y')ds, \\ \int_{x'}^{a} Q_2^\sigma(M)(s,y')f^+(x'-s,y')ds &=& Q_2^\sigma(M)(c_2,y')\int_{x'}^{a} f^+(x'-s,y')ds, \\ \int_0^{y'} Q_3^\sigma(M)(x',s)f^-(x',y'-s)ds &=& Q_3^\sigma(M)(x',c_3)\int_0^{y'} f^-(x',y'-s)ds, \\ \int_{y'}^{b} Q_4^\sigma(M)(x',s)f^-(x',y'-s)ds &=& Q_4^\sigma(M)(x',c_4)\int_{y'}^{b} f^-(x',y'-s)ds \end{array} \tag{3.10}$$

In the other hand in accordance with the Mean Value Theorem we can find $d_1 \in]0, x'[$, $d_2 \in]x', a[$, $d_3 \in]0, y'[$ and $d_4 \in]y', b[$ such that:

$$\begin{array}{rcl} \int_0^{x'} f^+(x'-s,y')ds &=& x'f^+(x'-d_1,y'), \\ \int_{x'}^{a} f^+(x'-s,y')ds &=& (a-x')f^+(x'-d_2,y'), \\ \int_0^{y'} f^-(x',y'-s)ds &=& y'f^-(x',y'-d_3), \\ \int_{y'}^{b} f^-(x',y'-s)ds &=& (b-y')f^-(x',y'-d_4) \end{array} \tag{3.11}$$

Hence letting $A^+(y') = \exp\left(\sigma \int_0^a \rho^+(M)(s-a,y')ds\right)$, $A^-(x') = \exp\left(\sigma \int_0^b \rho^-(M)(x',s-b)ds\right)$ we get:

$$\begin{array}{rcl} |N_1(x',y')| &\leqslant& |\phi_1'(y')| + |Q_1^\sigma(M)(c_1,y')|, \\ |N_2(x',y')| &\leqslant& |\phi_2'(y')|\left|A^+(y')\right| + |Q_2^\sigma(M)(c_2,y')|, \\ |N_3(x',y')| &\leqslant& |\phi_3'(x')| + |Q_3^\sigma(M)(x',c_3)|, \\ |N_4(x',y')| &\leqslant& |\phi_4'(x')|\left|A^-(x')\right| + |Q_4^\sigma(M)(x',c_4)| \end{array} \tag{3.12}$$

since $\left|g^\pm(x',y')\right| < 1$. From which we infer

$$\|\mathcal{T}_\sigma(M)\| \quad \leq \quad max\left(\|\phi_1'\|, \|\phi_2'\|\|A^+\|, \|\phi_3'\|, \|\phi_4'\|\|A^-\|\right) + \|Q^\sigma(M)\| \tag{3.13}$$

Thus $\mathcal{T}_\sigma$ is continuous and bounded since $A^\pm$, ϕ_i' and Q_i^σ, $i \in \Lambda$ are continuous and bounded. Hence if M is bounded then $N = \mathcal{T}_\sigma(M)$ is bounded since $\|\mathcal{T}_\sigma(M)\| \leq \|\mathcal{T}_\sigma\| \cdot \|M\|$.
Otherwise if N is the solution of (3.2) then $\forall\, i \in \Lambda$,

$$\left\{\begin{array}{rcl} \dfrac{\partial N_i}{\partial x'} + \sigma N_i\rho^+(M) &=& Q_i^\sigma(M), \quad i=1,2 \\ \dfrac{\partial N_i}{\partial y'} + \sigma N_i\rho^-(M) &=& Q_i^\sigma(M), \quad i=3,4 \end{array}\right.$$

Thus

$$\left\{\begin{array}{rcl} \dfrac{\partial N_i}{\partial x'} &=& Q_i^\sigma(M) - \sigma N_i\rho^+(M), \quad i=1,\ 2 \\ \dfrac{\partial N_i}{\partial y'} &=& Q_i^\sigma(M) - \sigma N_i\rho^+(M) \quad i=3,\ 4 \end{array}\right.$$

And

$$\left\{\begin{array}{rcl} \left|\dfrac{\partial N_i}{\partial x'}\right| &\leqslant& Q_i^\sigma(M) + \sigma N_i\rho^+(M) \quad i=1,2 \\ \left|\dfrac{\partial N_i}{\partial y'}\right| &\leqslant& Q_i^\sigma(M) + \sigma N_i\rho^+(M) \quad i=3,4 \end{array}\right.$$

Thus if M is bounded, $\dfrac{\partial N_i}{\partial x'}$ and $\dfrac{\partial N_i}{\partial y'}$ are uniformly bounded and it exists α and β such that

$$\left|\frac{\partial N_i}{\partial x'}\right| < \alpha \text{ in } [0,a]$$

and

$$\left|\frac{\partial N_i}{\partial y'}\right| < \beta \text{ in } [0, b]$$

Given $z_1 = (x'_1, y'_1) \in J$ and $z_2 = (x'_2, y'_2) \in J.$ We deduce from the Mean Value Theorem, that it exists $z_0 = (x'_0, y'_0) \in [z_1,\ z_2] \subset J$ such that

$$N_i(z_1) - N_i(z_2) \quad = \quad dN_i(z_0)(z_1 - z_2)$$

with

$$[z_1,\ z_2] = \left\{z \in \mathbb{R}^2 / z = t(z_1 - z_2) + z_2, t \in [0,1]\right\}$$

and

$$dN_i(z_0)(h) = \frac{\partial N_i}{\partial x'}(z_0)h_1 + \frac{\partial N_i}{\partial y'}(z_0)h_2 \quad \forall h = (h_1, h_2) \in \mathbb{R}^2$$

.

Hence

$$\begin{aligned} |N_i(z_1) - N_i(z_2)| &= |dN_i(z_0)(z_1 - z_2)| \\ &\leq \|dN_i(z_0)\| \|z_1 - z_2\| \end{aligned}$$

with

$$\begin{aligned} \|dN_i(z_0)\| &= \sup_{\|h\| \leq 1} \frac{|dN_i(z_0)|}{\|h\|} \\ &= \sup_{\|h\| \leq 1} \frac{\left|\frac{\partial N_i}{\partial x'}(z_0)h_1 + \frac{\partial N_i}{\partial y'}(z_0)h_2\right|}{|h_1| + |h_2|} \end{aligned}$$

But

$$\begin{aligned} \left|\frac{\partial N_i}{\partial x'}(z_0)h_1 + \frac{\partial N_i}{\partial y'}(z_0)h_2\right| &\leq \left|\frac{\partial N_i}{\partial x'}(z_0)\right| |h_1| + \left|\frac{\partial N_i}{\partial y'}(z_0)\right| |h_2| \\ &\leq max\left(\left|\frac{\partial N_i}{\partial x'}(z_0)\right|, \left|\frac{\partial N_i}{\partial y'}(z_0)\right|\right)(|h_1| + |h_2|) \\ Thus\ \frac{\left|\frac{\partial N_i}{\partial x'}(z_0)h_1 + \frac{\partial N_i}{\partial y'}(z_0)h_2\right|}{|h_1| + |h_2|} &\leq max\left(\left|\frac{\partial N_i}{\partial x'}(z_0)\right|, \left|\frac{\partial N_i}{\partial y'}(z_0)\right|\right) \\ &\leq max(\alpha, \beta) \end{aligned}$$

That is $\|dN_i(z_0)\| \leq max(\alpha, \beta)$. Then $|N_i(z_1) - N_i(z_2)| \leq max(\alpha, \beta)\|z_1 - z_2\|$. It is sufficient that $\|z_1 - z_2\| < \dfrac{\epsilon}{max(\alpha, \beta)}$ to have $|N_i(z_1) - N_i(z_2)| < \epsilon$ for all $i \in \Lambda$.

We prove that for all solution N of (3.2) :

$$\forall \epsilon > 0, \exists \xi > 0, \|z_1 - z_2\| < \xi \Rightarrow |N_i(z_1) - N_i(z_2)| < \epsilon \quad \forall\ z_1,\ z_2 \in J$$

The set of the solutions of (3.2) is thus equicontinuous so $\mathcal{T}_\sigma$ is compact on every bounded subset of $\mathscr{C}^4_+$. □

Lemma 3.4. *Every solution of the equation* $N = \lambda \mathcal{T}_\sigma(N)$, $0 < \lambda < 1$, *is bounded.*

Proof. N is a solution of $N = \lambda \mathcal{T}_\sigma(N)$ if and only if

$$\left\{\begin{array}{lcll} \dfrac{\partial N_1}{\partial x'} + \sigma N_1 \rho^+(N) & = & \lambda Q_1^\sigma(N) & (3.14.1)\\ \dfrac{\partial N_2}{\partial x'} + \sigma N_2 \rho^+(N) & = & \lambda Q_2^\sigma(N) & (3.14.2)\\ \dfrac{\partial N_3}{\partial y'} + \sigma N_3 \rho^-(N) & = & \lambda Q_3^\sigma(N) & (3.14.3)\\ \dfrac{\partial N_4}{\partial y'} + \sigma N_4 \rho^-(N) & = & \lambda Q_4^\sigma(N) & (3.14.4)\\ N_1(0,y') & = & \lambda \phi_1'(y') & (3.14.5)\\ N_2(a,y') & = & \lambda \phi_2'(y') & (3.14.6)\\ N_3(x',0) & = & \lambda \phi_3'(x') & (3.14.7)\\ N_4(x',b) & = & \lambda \phi_4'(x') & (3.14.8) \end{array}\right. \tag{3.14}$$

As the partial macroscopic densities $\rho^+(N)$ and $\rho^-(N)$ are conserved for the Broadwell model, making the sums (3.14.1) + (3.14.2) and (3.14.3) + (3.14.4) , we obtain for their determination the following system of partial differential equations:

$$\left\{\begin{array}{lcll} \frac{\partial\left[\rho^+(N)\right]}{\partial x'} + (1-\lambda)\sigma\left[\rho^+(N)\right]^2 & = & 0 & (3.15.1)\\ \frac{\partial\left[\rho^-(N)\right]}{\partial y'} + (1-\lambda)\sigma\left[\rho^-(N)\right]^2 & = & 0 & (3.15.2) \end{array}\right. \tag{3.15}$$

The unique solution of the system (3.15) is obviously

$$\left\{\begin{array}{lcl} \rho^+(x',y') & = & \dfrac{1}{(1-\lambda)\sigma x' + h^+(y')}\\ \\ \rho^-(x',y') & = & \dfrac{1}{(1-\lambda)\sigma y' + h^-(x')} \end{array}\right. \tag{3.16}$$

The problem (3.14) is a two point boundary value problem and only a part of the data are given at each boundary namely $N_1(0,y')$ on the line $x'=0$, $N_2(a,y')$ on the line $x'=a$, $N_3(x',0)$ on the line $y'=0$ and $N_4(x',b)$ on the line $y'=b$. We thus introduce the positive functions of y', α_k^+ and the positive functions of x', α_k^-, $k=0,1$ such that

$$\begin{array}{lcl} N_2(0,y') & = & \alpha_0^+(y')N_1(0,y')\\ N_1(a,y') & = & \alpha_1^+(y')N_2(a,y')\\ N_4(x',0) & = & \alpha_0^-(x')N_3(x',0)\\ N_3(x',b) & = & \alpha_1^-(x')N_4(x',b) \end{array} \tag{3.17}$$

We emphasize the fact that the relations (3.17) are by no means reflection laws and are obtained merely by comparing functions of the same variables at the boundaries of the rectangle J and consequently are general. In the particular case of impermeable boundaries , the vanishing of the normal velocity on each boundary yields the relations

$$\begin{array}{lcll} N_2(0,y') & = & N_1(0,y') \text{ at } x'=0 & \forall y' \in [0,b]\,,\\ N_1(a,y') & = & N_2(a,y') \text{ at } x'=a & \forall y' \in [0,b]\,,\\ N_4(x',0) & = & N_3(x',0) \text{ at } y'=0 & \forall x' \in [0,a]\,,\\ N_3(x',b) & = & N_4(x',b) \text{ at } y'=b & \forall x' \in [0,a]\,. \end{array} \tag{3.18}$$

which amount to take $\alpha_k^+(y') = 1$ and $\alpha_k^-(x') = 1$ $\forall (x',y') \in J$, $k=0,1$.
With the relations (3.17) we can compute the values of $\rho^\pm$ at the boundaries from which we get:

$$\begin{array}{lcl} h^+(y') = \dfrac{1}{\left[1+\alpha_0^+(y')\right]\lambda\phi_1'(y')} & = & \dfrac{1}{\left[1+\alpha_1^+(y')\right]\lambda\phi_2'(y')} + \sigma(\lambda-1)\\ h^-(x') = \dfrac{1}{\left[1+\alpha_0^-(x')\right]\lambda\phi_3'(x')} & = & \dfrac{1}{\left[1+\alpha_1^-(x')\right]\lambda\phi_4'(x')} + \sigma(\lambda-1) \end{array} \tag{3.19}$$

from the systems (3.16) and (3.19) we deduce

$$\left\{\begin{array}{lcl}\rho^{+}\left(x',y'\right) & = & \dfrac{1}{(1-\lambda)\sigma x' + \dfrac{1}{\left[1+\alpha_0^{+}\left(y'\right)\right]\lambda\phi_1'(y')}} \\ \rho^{-}\left(x',y'\right) & = & \dfrac{1}{(1-\lambda)\sigma y' + \dfrac{1}{\left[1+\alpha_0^{-}\left(x'\right)\right]\lambda\phi_3'(x')}}\end{array}\right. \tag{3.20}$$

Thus for $0 < \lambda < 1$, ρ^{+} and ρ^{-} are continuous and bounded as ϕ_i', $i = 1,3$ and $\alpha_k^{\pm}$, $k = 0,1$. The mean density is thus bounded and so are the number densities N_i,$\forall i \in \Lambda$.

We point out the fact that for $\lambda = 1$ the solutions ρ^{+} and ρ^{-} of (3.20) are not singular and moreover satisfy the conservation equations of the partial macroscopic densities. Accordingly they depend upon one variable. □

Finally we conclude to the existence of solution of problem (3.1) by using the fixed point theorem of Schaefer Smart (1974):

Theorem 3.5. *Let T be a continuous and compact mapping of a Banach space X into itself, such that the set $\{x' \in X', x' = \lambda T(x')\}$ is bounded $\forall\, \lambda,\ 0 < \lambda < 1$. Then T has a fixed point.*

4 Exact solutions of (3.1)

For $\lambda = 1$, the densities (3.20) are solutions of the conservation equations of the partial mean densities $\rho^{\pm}$ of the model . Hence ρ^{+} and ρ^{-} are known and we have :
$\rho^{+}\left(N\right)\left(y'\right) = \left(N_1 + N_2\right)\left(y'\right)$, $\rho^{-}\left(N\right)\left(x'\right) = \left(N_3 + N_4\right)\left(x'\right)$.
Then

$$\left\{\begin{array}{lcl} N_2(x',y') & = & \rho^{+}(y') - N_1(x',y') \\ N_4(x',y') & = & \rho^{-}(x') - N_3(x',y')\end{array}\right. \tag{4.1}$$

and the system (3.1) becomes :

$$\left\{\begin{array}{lclcl} \dfrac{\partial N_1}{\partial x'} & = & s\left[\left(N_1 - \dfrac{\rho^{+}}{2}\right)^2 - \left(N_3 - \dfrac{\rho^{-}}{2}\right)^2 + \dfrac{\rho^{-2} - \rho^{+2}}{4}\right] & = & Q_1(N) \\ \dfrac{\partial N_3}{\partial y'} & = & -s\left[\left(N_1 - \dfrac{\rho^{+}}{2}\right)^2 - \left(N_3 - \dfrac{\rho^{-}}{2}\right)^2 + \dfrac{\rho^{-2} - \rho^{+2}}{4}\right] & = & -Q_1(N) \\ N_2(x',y') & = & \rho^{+}(y') - N_1(x',y') & & \\ N_4(x',y') & = & \rho^{-}(x') - N_3(x',y') & & \\ N_1(0,y') & = & \phi_1'(y') & & \\ N_2(a,y') & = & \phi_2'(y') & & \\ N_3(x',0) & = & \phi_3'(x') & & \\ N_4(x',b) & = & \phi_4'(x') & & \end{array}\right. \tag{4.2}$$

The boundary value problem for the microscopic densities N_i, $i = 1,3$ is thus:

$$\left\{\begin{array}{l} \dfrac{\partial N_1}{\partial x'} = -\dfrac{\partial N_3}{\partial y'} = s\left[\left(N_1 - \dfrac{\rho^{+}}{2}\right)^2 - \left(N_3 - \dfrac{\rho^{-}}{2}\right)^2 + \dfrac{\rho^{-2} - \rho^{+2}}{4}\right] = Q_1(N) \\ N_1(0,y') = \phi_1'(y') \\ N_1(a,y') = \rho^{+}(y') - \phi_2'(y') \\ N_3(x',0) = \phi_3'(x') \\ N_3(x',b) = \rho^{-}(x') - \phi_4'(x') \end{array}\right. \tag{4.3}$$

Letting $F_1(x',y') = N_1(x',y') - \frac{\rho^+}{2}(y')$, $F_3(x',y') = N_3(x',y') - \frac{\rho^-}{2}(x')$ the system (4.3) take the form:

$$\begin{cases} \frac{\partial F_1}{\partial x'} = -\frac{\partial F_3}{\partial y'} = s\left(F_1^2 - F_3^2 + \frac{\rho^{-2} - \rho^{+2}}{4}\right) = Q_1(F) \\ F_1(0,y') = \frac{\rho^+}{2}(y') - \phi_1'(y') \\ F_1(a,y') = \phi_2'(y') - \frac{\rho^+(y')}{2} \\ F_3(x',0) = \frac{\rho^-(x')}{2} - \phi_3'(x') \\ F_3(x',b) = \phi_4'(x') - \frac{\rho^-(x')}{2} \end{cases} \tag{4.4}$$

. The system (4.4) has a simpler form but its exact resolution is complicated. However it permits to find exact solutions of the problem (4.2) in particular cases.

4.1 Maxwellian solutions

An obvious solution of system (4.4) is $F_1(x',y') = \frac{\rho^+(y')}{2}$ and $F_3(x',y') = \frac{\rho^-(x')}{2}$ which leads to $\phi_1'(y') = \rho^+(y')$, $\phi_3'(x') = \rho^-(x')$, $Q_1(F) = 0$ and $\phi_2' = \phi_4' = 0$. Hence the microscopic densities N_2 and N_4 are zero and the model is reduced to a two velocity model. This solution is clearly unphysical. The solution $F_1(x',y') = \frac{1}{2}\sqrt{\rho^{+2}(y') - 4c_1}$, $F_3(x',y') = \frac{1}{2}\sqrt{\rho^{-2}(x') - 4c_1}$ for $c_1 \geqslant 0$ is also a maxwellian solution. Hence:

$$\begin{aligned} N_1(x',y') &= \frac{\rho^+(y')}{2} + \frac{1}{2}\sqrt{\rho^{+2}(y') - 4c_1} \\ N_2(x',y') &= \frac{\rho^+(y')}{2} - \frac{1}{2}\sqrt{\rho^{+2}(y') - 4c_1} \\ N_3(x',y') &= \frac{\rho^-(x')}{2} + \frac{1}{2}\sqrt{\rho^{-2}(x') - 4c_1} \\ N_4(x',y') &= \frac{\rho^-(x')}{2} - \frac{1}{2}\sqrt{\rho^{-2}(x') - 4c_1} \end{aligned} \tag{4.5}$$

Taking into account the boundary conditions, we get:

$$\begin{aligned} &\rho^+(y') = \phi_1'(y') + \phi_2'(y') \\ &\rho^-(x') = \phi_3'(x') + \phi_4'(x') \\ &c_1 = \phi_1'(y')\phi_2'(y') = \phi_3'(x')\phi_4'(x') \end{aligned} \tag{4.6}$$

The validity of the third relation (4.6) imposes the dependence of the boundary data in the form:

$$\begin{aligned} \phi_2'(y') &= \tfrac{c_1}{\phi_1'(y')} \\ \phi_4'(x') &= \tfrac{c_1}{\phi_3'(x')} \end{aligned} \tag{4.7}$$

The Maxwellian solutions are thus:

$$\begin{aligned} N_1(x',y') &= \phi_1'(y') \\ N_2(x',y') &= \tfrac{c_1}{\phi_1'(y')} \\ N_3(x',y') &= \phi_3'(x') \\ N_4(x',y') &= \tfrac{c_1}{\phi_3'(x')} \\ {\phi_1'}^2 > c_1 \quad &, \quad {\phi_3'}^2 > c_1 \end{aligned} \tag{4.8}$$

The solutions (4.8) are associated to the macroscopic variables:

$$\begin{array}{rcl} \rho & = & \phi_1' + \phi_3' + \frac{c_1}{\phi_1'} + \frac{c_1}{\phi_3'} \\ \rho U & = & c(\phi_1' - \frac{c_1}{\phi_1'}) \\ \rho V & = & c(\phi_3' - \frac{c_1}{\phi_3'}) \end{array} \tag{4.9}$$

So they are merely particular expressions of the unique maxwellian solutions of the model associated to the macroscopic variables ρ, U and V defined by:

$$\begin{array}{rcl} N_{1M} & = & \frac{\rho}{4}\,(1+u+v)\,(1+u-v) \\ N_{2M} & = & \frac{\rho}{4}\,(1-u-v)\,(1-u+v) \\ N_{3M} & = & \frac{\rho}{4}\,(1+u+v)\,(1-u+v) \\ N_{4M} & = & \frac{\rho}{4}\,(1-u-v)\,(1+u-v) \\ u=\dfrac{U}{c} & , & v=\dfrac{V}{c} \end{array} \tag{4.10}$$

4.2 Non maxwellian solutions

For ρ^+, ρ^-, k and l constant such that $k\rho^- - l\rho^+ \neq 0$ a solution of (4.2) is given by:

$$\begin{array}{rcl} N_1(x',y') & = & \dfrac{k}{M(x',y')} \\ N_3(x',y') & = & \dfrac{l}{M(x',y')} \\ M(x',y') & = & c_0 \exp\left[2s\left(k\rho^- - l\rho^+\right)\left(\dfrac{x'}{k} - \dfrac{y'}{l}\right)\right] + \dfrac{k^2 - l^2}{k\rho^- - l\rho^+} \end{array} \tag{4.11}$$

This solution is non maxwellian whatever ρ^+, ρ^-, k and l when $k\rho^- - l\rho^+ \neq 0$ as c_0 is a non zero scaling parameter. Moreover when $\rho^+ = \rho^- = \rho$ constant we have a solution given by:

$$\begin{array}{rcl} N_1(x',y') & = & \dfrac{\rho}{2} + \dfrac{c_2 c_3}{2s\left(c_2^2 - c_3^2\right)}\left[-c_2 + c_3 \tanh\left(c_1 + c_2 x' + c_3 y'\right)\right] \\ N_3(x',y') & = & \dfrac{\rho}{2} + \dfrac{c_2 c_3}{2s\left(c_2^2 - c_3^2\right)}\left[c_3 - c_2 \tanh\left(c_1 + c_2 x' + c_3 y'\right)\right] \end{array} \tag{4.12}$$

The fact that we have a maxwellian and two non maxwellian solutions for constant ρ^+ and ρ^- shows the non uniqueness of the solutions of the system (4.2) in general.

5 Steady flow in box

We investigate in this section the flow of a discrete gas in a box in order to compute accommodation coefficients. In the statement of a flow problem, in contrast to the boundary value problem (2.2) in which they are assumed known, the boundary conditions ϕ_i' depend upon the accommodation coefficients which describe the interactions between the particles of the gas and those of the boundaries of the flow domain. The accommodation coefficients are unknowns of the problem and classically one has to prescribe reflection laws to get additional relations for their determination which is achieved only when the mathematical problem is solved [Cercignani (1969); d'Almeida and Gatignol (1995); Gatignol (1977)].
We choose the reference quantities $n_0 = 2\sqrt{c_1}$ and a respectively for the densities and the lenghts and introduce the following dimensionless variables and parameters:

$$x = \tfrac{x'}{a},\ y = \tfrac{y'}{a},\ \varepsilon = \tfrac{b}{a},\ K_n = (sn_0 a)^{-1},\ \widetilde{N}_i = \tfrac{N_i}{n_0},\ \phi_i = \tfrac{\phi_i'}{n_0},\ \widetilde{Q} = \tfrac{Q}{sn_0^2}. \tag{5.1}$$

The problem (2.2) is put in the nondimensional form:

$$\left\{\begin{array}{lcl}\dfrac{\partial \widetilde{N}_1}{\partial x} &=& \frac{2}{K_n\varepsilon}\left(\widetilde{N}_3\widetilde{N}_4-\widetilde{N}_1\widetilde{N}_2\right)=\widetilde{Q}(\widetilde{N})\\ \dfrac{\partial \widetilde{N}_2}{\partial x} &=& -\widetilde{Q}(\widetilde{N})\\ \dfrac{\partial \widetilde{N}_3}{\partial y} &=& -\varepsilon\widetilde{Q}(\widetilde{N})\\ \dfrac{\partial \widetilde{N}_4}{\partial y} &=& \varepsilon\widetilde{Q}(\widetilde{N})\\ \widetilde{N}_1(0,y) &=& \phi_1(y)\\ \widetilde{N}_2(1,y) &=& \phi_2(y)\\ \widetilde{N}_3(x,0) &=& \phi_3(x)\\ \widetilde{N}_4(x,\varepsilon) &=& \phi_4(x)\end{array}\right. \tag{5.2}$$

The dimensionless macroscopic variables of the flow are the mean density ρ, the tangential velocity u and the transversal velocity v given by:

$$\begin{array}{lclcl}\rho &=& \widetilde{N}_1+\widetilde{N}_2+\widetilde{N}_3+\widetilde{N}_4 &=& \rho^+(y)+\rho^-(x)\\ \rho u &=& \widetilde{N}_1-\widetilde{N}_2 &=& 2\widetilde{N}_1-\rho^+(y)\\ \rho v &=& \widetilde{N}_3-\widetilde{N}_4 &=& 2\widetilde{N}_3-\rho^-(x)\end{array} \tag{5.3}$$

The Maxwellian densities of the model associated with the dimensionless macroscopic variables ρ, u and v are:

$$\begin{array}{lcl}\widetilde{N}_{1M} &=& \dfrac{\rho}{4}(1+u+v)(1+u-v)\\ \widetilde{N}_{2M} &=& \dfrac{\rho}{4}(1-u-v)(1-u+v)\\ \widetilde{N}_{3M} &=& \dfrac{\rho}{4}(1+u+v)(1-u+v)\\ \widetilde{N}_{4M} &=& \dfrac{\rho}{4}(1-u-v)(1+u-v)\end{array} \tag{5.4}$$

The microscopic densities of the discrete gas in Maxwellian equilibrium with a wall are the maxwellian densities associated with 1, the tangential and transversal velocities of the wall. Assume that the macroscopic velocity of the box is $\vec{U_w}=(u_w(x,y),v_w(x,y))$. The microscopic densities of the gas in maxwellian equilibrium with the box are:

$$\begin{array}{lcl}\widetilde{N}_{1M} &=& \dfrac{1}{4}(1+u_w+v_w)(1+u_w-v_w)\\ \widetilde{N}_{2M} &=& \dfrac{1}{4}(1-u_w-v_w)(1-u_w+v_w)\\ \widetilde{N}_{3M} &=& \dfrac{1}{4}(1+u_w+v_w)(1-u_w+v_w)\\ \widetilde{N}_{4M} &=& \dfrac{1}{4}(1-u_w-v_w)(1+u_w-v_w)\end{array} \tag{5.5}$$

It is usually assumed when the exchanges of mass or energy of a gas and its surrounding only result from the collisions of its particles with its boundaries that only the microscopic densities of the reflected particles are known near the walls [Cercignani (1969)]. We can compare these densities to those of the fictituous gas in equlibrium with each wall and introduce the functions $l_i(y)$, $i=1,2$ and $l_j(x)$, $j=3,4$ such that:

$$\begin{array}{lcl}\widetilde{N}_1(0,y) &=& \dfrac{l_1(y)}{4}(1+u_w(0,y)+v_w(0,y))(1+u_w(0,y)-v_w(0,y))\\ \widetilde{N}_2(1,y) &=& \dfrac{l_2(y)}{4}(1-u_w(1,y)-v_w(1,y))(1-u_w(1,y)+v_w(1,y))\\ \widetilde{N}_3(x,0) &=& \dfrac{l_3(x)}{4}(1+u_w(x,0)+v_w(x,0))(1-u_w(x,0)+v_w(x,0))\\ \widetilde{N}_4(x,\varepsilon) &=& \dfrac{l_4(x)}{4}(1-u_w(x,\varepsilon)-v_w(x,\varepsilon))(1+u_w(x,\varepsilon)-v_w(x,\varepsilon))\end{array} \tag{5.6}$$

Using the form (4.8) of the maxwellian solutions (4.5) we have:

$$\begin{aligned} \widetilde{N}_1(0,y) &= \phi_1(y) \\ \widetilde{N}_2(1,y) &= \frac{1}{4\phi_1(y)} \\ \widetilde{N}_3(x,0) &= \phi_3(x) \\ \widetilde{N}_4(x,\varepsilon) &= \frac{1}{4\phi_3(x)} \end{aligned} \tag{5.7}$$

We can thus explicitly determine the functions $l_k,\ k=1,3$ which are given by:

$$\begin{aligned} l_1(y) &= \frac{4\phi_1(y)}{[1+u_w(0,y)+v_w(0,y)]\,[1+u_w(0,y)-v_w(0,y)]} \\ l_2(y) &= \frac{1}{\phi_1(y)\,[1-u_w(1,y)-v_w(1,y)]\,[1-u_w(1,y)+v_w(1,y)]} \\ l_3(x) &= \frac{4\phi_3(x)}{[1+u_w(x,0)+v_w(x,0)]\,[1-u_w(x,0)+v_w(x,0)]} \\ l_4(x) &= \frac{1}{\phi_3(x)\,[1-u_w(x,\varepsilon)-v_w(x,\varepsilon)]\,[1+u_w(x,\varepsilon)-v_w(x,\varepsilon)]} \end{aligned} \tag{5.8}$$

We introduce now reflection laws. We prescribe that particles of opposite velocities are reflected with the same accommodation coefficients. That is:

$$\begin{aligned} l_1(y) &= l_2(y),\ \forall y \in [0,\varepsilon] \\ l_3(x) &= l_4(x),\ \forall x \in [0,1] \end{aligned} \tag{5.9}$$

We infer from these additional relations:

$$\begin{aligned} \phi_1(y) &= \frac{1}{2}\sqrt{\frac{\left\{[1+u_w(0,y)]^2-v_w(0,y)^2\right\}}{\left\{[1-u_w(1,y)]^2-v_w(1,y)^2\right\}}} \\ \phi_3(x) &= \frac{1}{2}\sqrt{\frac{\left\{[1+v_w(x,0)]^2-u_w(x,0)^2\right\}}{\left\{[1-v_w(x,\varepsilon)]^2-u_w(x,\varepsilon)^2\right\}}} \\ l_1(y) &= \frac{1}{\sqrt{\left\{[1+u_w(0,y)]^2-v_w(0,y)^2\right\}\left\{[1-u_w(1,y)]^2-v_w(1,y)^2\right\}}} \\ l_3(x) &= \frac{1}{\sqrt{\left\{[1+v_w(x,0)]^2-u_w(x,0)^2\right\}\left\{[1-v_w(x,\varepsilon)]^2-u_w(x,\varepsilon)^2\right\}}}. \end{aligned} \tag{5.10}$$

The relations (5.11) give the boundary data ϕ_j in terms of the macroscopic variables of the box's walls. In fact the walls do not move freely as we assume in our computations. Thus when we take into account the fact that for a solid box all the walls have the same constant velocity we have :

$$\begin{aligned} \phi_1 &= \frac{1}{2}\sqrt{\frac{[1+u_w+v_w][1+u_w-v_w]}{[1-u_w-v_w][1-u_w+v_w]}} \\ \phi_3 &= \frac{1}{2}\sqrt{\frac{[1+u_w+v_w][1-u_w+v_w]}{[1-u_w-v_w][1+u_w-v_w]}} \\ l_1 &= \frac{1}{\sqrt{[1+u_w+v_w][1+u_w-v_w][1-u_w-v_w][1-u_w+v_w]}} \\ l_3 &= \frac{1}{\sqrt{[1+u_w+v_w][1-u_w+v_w][1-u_w-v_w][1+u_w-v_w]}}. \end{aligned} \tag{5.11}$$

The accommodation coefficients are equal although the boundary conditions are different in this more realistic case. Hence the microscopic densities of the reflected particles at the wall are proportional to the microscopic densities of the fictitous gas in equilibrium with the wall corresponding to those particles. As the coefficient of proportionality is the same we have the diffuse reflection law of interaction [Cercignani (1969); d'Almeida and Gatignol (1995)].

6 CONCLUSIONS

We show that the boundary value problem for the two dimensional Broadwell model has a bounded solution . Only positivity and boundedness are assumed for the data. The solution is not unique.

Some exact analytic solutions are built. An application to the determination of the accommodation coefficients on the boundaries of a gas flow in a box is performed. Exact analytic expressions of the accommodation coefficients are given and the diffuse reflection law is obtained in a particular case. The method is simple and it will be interesting to check its applicability to more complex discrete models.

References

Broadwell, J. E. (1964). Shock structure in a simple discrete velocity gas. Phys. Fluids 7, 1243-1247.

Gatignol, R. (1977). Kinetic theory boundary conditions for discrete velocity gases. Phys. Fluids., 20, 2022-2030.

Cercignani, C. (1969). Mathematical methods in kinetic theory. Plenum Press.

d'Almeida, A. (2007). Exact solutions for discrete velocity models. Mechanic Research Communications 34, 405-409.

d'Almeida, A. (2008). Transition of unsteady flows to steady state in the process of evaporation and condensation. CRMécanique 336, 612-615.

d'Almeida, A. and Gatignol, R. (1995). Boundary conditions for discrete models of gases and applications to Couette flows. In: Computational Fluid Dynamics, (publisher: D. Leutloff, D. and Srivastava, R. C.), Eds,Springer-Verlag, 115-130.

d'Almeida, A. and Gatignol, R. (2003). The half space problem in discrete kinetic theory. Mathematical Models and Methods in Applied Sciences, 13, 99-119.

Cercignani, C., Illner, R., Shinbrot, M. (1988). A boundary value problem for the two dimensional broadwell model. Commun. Math. Phys. 114, 687-698.

Cabannes, H. (1980). The Discrete Boltzmann Equation (Theory and Applications).

Lecture Notes, Spring Quarter .

Cornille, H. and d'Almeida, A. (2002). Temperature and pressure criteria for half- space discrete velocity models. Eur. J. Mech. Fluids B 21, 355-370.

Natta, T., Agosseme, K. A. and d'Almeida, A. (2018). Existence and uniqueness of solution of the ten discrete velocity model C_1. JAMCS 29, 1-12.

Platkowski, T. and Illner, R. (1988). Discrete velocity models of the Boltzmann equation: a survey on the mathematical aspects of the theory. SIAM Review, 213-255.

Smart, D. R. (1974). Fixed Point Theorems. Cambridge University Press, New York .