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Some exact solutions of compressible and incompressible Euler equations

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Abstract

In this paper, we use a surprised system to construct some exact solutions of compressible Euler equations with two and three dimension. Furthermore, we also give other exact solutions of three dimension incompressible Euler equations.

Keywords: compressible and incompressible Euler equations; cylindrical coordinate; Exact solutions.

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1 Introduction

In this paper, we are concerned with the exact solution to 2 and 3 dimension compressible Euler equations

$$\begin{cases} \partial_t \rho + div(\rho \bar{u}) = 0 \\ \partial_t(\rho \bar{u}) + div(\rho \bar{u} \otimes \bar{u}) + \nabla \bar{P}(\rho) = 0 \\ \bar{u}|_{t=0} = \bar{u}_0, \rho|_{t=0} = \rho_0 \end{cases} \tag{1.1}$$

where $(t, x) \in R^+ \times R^n$, $n = 2, 3$, and $\bar{u} \in R^n$, ρ, $\bar{P}$ stand for the velocity, density,pressure of gases respectively. If $(\rho, \bar{u}) \in C^1$ is a solution of the above systems with $\rho \neq 0$, then it admits the following equivalent form

$$\begin{cases} \partial_t \rho + div(\rho \bar{u}) = 0 \\ \partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \bar{p}(\rho) = 0 \\ \bar{u}|_{t=0} = \bar{u}_0, \rho|_{t=0} = \rho_0 \end{cases} \tag{1.2}$$

The Euler equations are a very important model in fluid mechanics, which have been wide used in many areas. In the mathematical theory, there are much work, such as the local solution's existence. However, about the system, there are still difficult problems unsolved. For example, the global existence is opened. In the history researching the Euler equations (1.1), scholars prefer to demonstrate the polytropic gases, namely $P = \kappa \rho^\gamma, \kappa > 0, \gamma > 1.$

Under the case, we can get great deal of results by search tools, but in some ways, the local existence is still opened. Whatever the pressure is, the significance consequence is little. In order to obtain valuable result, constructing their explicit solutions is a very significance part in mathematical physics. Exact solutions can provide the concrete examples to understand their nonlinear phenomena and physical applications. In addition, there some works about the exact solutions of (1.2), such as [1,2,3,8]. In [5], K.L. Cheun gave some blow-up solutions. In our paper, we also give some blow-up exact solutions by choosing suitable parametric functions. And Blake in [4] gave periodic structure's solutions. The same solution in our work, is given. Moreover, people also consider other solution with befitting conditions, for example [6,7].

We don't directly study (1.1) or (1.2), but demonstrate the surprising systems

$$\begin{cases} \partial_t divu + u\cdot\nabla divu = \varepsilon(divu)^2 + f(t) \\ \partial_t u + u\cdot\nabla u + \nabla p = 0 \\ u|_{t=0} = u_0,\ divu|_{t=0} = divu_0 \end{cases} \tag{1.3}$$

where $f(t)$ is a function depending only on time t, and $\varepsilon = \pm 1$. Using the above equations's solutions, we can construct the solutions belonging to (1.2), and at same time, we also give some exact solutions of the 3 dimension incompressible Euler quations

$$\begin{cases} \partial_t \tilde{u} + \tilde{u}\cdot\nabla\tilde{u} + \nabla\tilde{p} = 0 \\ div\tilde{u} = 0 \\ \tilde{u}|_{t=0} = \tilde{u}_0 \end{cases} \tag{1.4}$$

In fact, when $\varepsilon = -1$ and $f(t) \equiv 0$, if u, p is a solution of (1.3) with $divu \in C^1$, then by the the inverse function theorem, we have $p(u) = p\circ(div)^{-1}(divu) = \bar{p}(divu)$. That is,

$$\bar{u} = u, \rho = divu, \bar{p} = p\circ(div)^{-1} \tag{1.5}$$

is a solution of (1.2). While, $\varepsilon = 1$, (1.4) have a family solutions likely

$$\tilde{u}(t,\tilde{x}) = (u, x_{n+1}divu), \tilde{p}(t,\tilde{x}) = p(t,x) + \frac{1}{2}x_{n+1}^2 f(t) \tag{1.6}$$

So,(1.3) is a magical system.

In this paper, we considerate (1.3) with 2 and 3 dimension under cylindrical coordinate. Since 2 dimension cylindrical coordinate is contained in the 3 dimension, so we state to study (1.3) with 3 dimension. We demonstrate the axisymmetric solution

$$u(t,r,z) = u^r(t,r,z)\dot{e}_r + u^\theta(t,r,z)\dot{e}_\theta + u^z(t,r,z)\dot{e}_z \tag{1.7}$$

with

$$\dot{e}_r = \left(\frac{x}{r}, \frac{y}{r}, 0\right), \dot{e}_\theta = \left(-\frac{y}{r}, \frac{x}{r}, 0\right), \dot{e}_z = (0,0,1), r = \sqrt{x_1^2 + x_2^2}$$

Then,by calculating, we get the facts that

$$u\cdot\nabla = u^r\partial_r + \frac{1}{r}u^\theta\partial_\theta + u^z\partial_z,$$

and

$$divu = \frac{1}{r}u^r + u^r_r + u^z_z$$

Thus, we can reduce the axisymmetric equations

$$\begin{cases} \left(\frac{1}{r}u^r + u^r_r + u^z_z\right)_t + u^r\left(\frac{1}{r}u^r + u^r_r + u^z_z\right)_r + u^z\left(\frac{1}{r}u^r + u^r_r + u^z_z\right)_z = \varepsilon\left(\frac{1}{r}u^r + u^r_r + u^z_z\right)^2 + f(t) \\ u^r_t + u^r u^r_r - \frac{1}{r}(u^\theta)^2 + u^z u^r_z + p_r = 0 \\ u^\theta_t + u^r u^\theta_r + \frac{1}{r}u^r u^\theta + u^z u^\theta_z = 0 \\ u^z_t + u^r u^z_r + u^z u^z_z + p_z = 0 \end{cases} \tag{1.8}$$

In the following, we apply (1.8) to construct some exact solutions of Euler equations.

2 The exact solutions for $n = 2$

Under the case, (1.7) and (1.8) respectively becomes

$$u(t,r) = u^r(t,r)\dot{e}_r + u^\theta(t,r)\dot{e}_\theta \tag{2.1}$$

with $\dot{e}_r = (\frac{x}{r}, \frac{y}{r})$, $\dot{e}_\theta = (-\frac{y}{r}, \frac{x}{r})$, and

$$\begin{cases} (\frac{1}{r}u^r + u^r_r)_t + u^r(\frac{1}{r}u^r + u^r_r)_r = \varepsilon(\frac{1}{r}u^r + u^r_r)^2 + f(t) \\ u^r_t + u^r u^r_r - \frac{1}{r}(u^\theta)^2 + p_r = 0 \\ u^\theta_t + u^r u^\theta_r + \frac{1}{r}u^r u^\theta = 0 \end{cases} \tag{2.2}$$

Based on the above system, we know that once we have the expression of u^r, we right now use the characteristics' method to give exact expression of u^θ depending on the equation

$$u^\theta_t + u^r u^\theta_r + \frac{1}{r}u^r u^\theta = 0 \quad (2.3)$$

and also have

$$p = \int \frac{1}{r}(u^\theta)^2 dr - \int u^r dr_t - \frac{1}{2}(u^r)^2 \tag{2.4}$$

Thus, we employ the first equation in (2.2) to give some exactly type of u^r.

Write

$$\eta = \frac{1}{r}u^r + u^r_r$$

Then, by ODE theorem we have

$$u^r = \frac{e(t)}{r} + \frac{1}{r}\int r\eta dr$$

Applying the expression and the first equation in (2.2), we get

$$\eta_t + (\frac{e(t)}{r} + \frac{1}{r}\int r\eta dr)\eta_r = \varepsilon\eta^2 + f(t)$$

Let

$$\eta(t, r) = w(t, z),\ z = r^2,$$

then, we obtain

$$w_t + (2e(t) + \int wdz)w_z = \varepsilon w^2 + f(t).\ \xi =$$

Against, writing

$$2e(t) + \int wdz,$$

then, in the end, we think about the problem

$$\xi_{zt} + \xi\xi_{zz} = \varepsilon\xi_z^2 + f(t) \quad (2.5)$$

We learn the case $\varepsilon = 1$. We build up the type solution

$$\xi(t, z) = \theta(t) + z\pi(t) + p(t)e^{zk(t)} + q(t)e^{-zk(t)}$$

After computing, we get

$$\xi(t,z) = \theta(t) + z\pi(t) + \alpha \exp\left(\int\int 3\pi(t) - \theta(t)k(t)dt + zk(t)\right) + \beta \exp\left(\int\int 3\pi(t) + \theta(t)k(t)dt - zk(t)\right)$$

and

$$f(t) = 4\alpha\beta s^2 \exp\left(4\int\int \pi(t)dt\right) + g(t,\pi), \alpha, \beta, s \in \mathbb{R},$$

with $k(t) = s\exp(-\int\pi(t)dt)$, where $\pi(t)$ satisfies the ODE:

$$\pi' = \pi^2 + g(t,\pi), g(t,0) = 0.$$

If $\pi \equiv 0$, then $f(t) = 4\alpha\beta s^2$. Thanks to the arbitrary of α, β, s, we require that $f(t)$ is arbitrary real number. Thus, we have

$$u^r(t,r) = \frac{e(t)}{r} + \frac{r\pi(t)}{2} + \frac{\alpha}{2r}\exp\left(\int\int 3\pi(t) - \theta(t)k(t)dt + r^2k(t)\right) + \frac{\beta}{2r}\exp\left(\int\int 3\pi(t) + \theta(t)k(t)dt - r^2k(t)\right)$$

At the same time, we also have periodic solution $\xi(t,z)$ due to $\theta(t)$, namely that

$$\xi(t,z) = \theta(t) + k\cos(\alpha z - \alpha\int\theta(t)dt + \beta) + k\sin(\alpha z - \alpha\int\theta(t)dt + \beta)$$ where $f(t) = -2\alpha^2k^2, \alpha, \beta, k \in \mathbb{R}$. At once, we get other type

$$u^r(t,r) = \frac{e(t)}{r} + \frac{\sqrt{2}k}{2r}\sin\left(\int \alpha r^2 - \alpha\int\theta(t)dt + \beta + \frac{\pi}{4}\right)$$

what is more, if $\xi(t,x)$ is a solution, then the function

$$\lambda^a\xi(\lambda^{a+b}t, \lambda^b x) \text{ is}$$

also a solution with $\lambda, a, b \in \mathbb{R} - \{0\}$ and $\lambda^{2a+2b}f(t)$.

When $\varepsilon = -1$ and $f(t) \equiv 0$, (2.5) reduces

$$\begin{cases}\xi_t + \xi\xi_z = h(t) \\ \xi(0,z) = g(z)\end{cases} \tag{2.6}$$

We use characteristics of the method to build solution. Hence, we have

$$\xi(t,z) = \tilde{\xi}(t,z_0) = g(z_0) + \int_0^t h(t')dt'$$

where z_0 meets

$$z = z_0 + tg(z_0) + \int_0^t\int_0^{t'} h(t'')dt''dt'$$

Choosing suitable $g(z_0)$,so that $G(z_0) = z_0 + tg(z_0) \in C^1$, then we have

$$\xi(t,z) = g\left(G^{-1}\left(z - \int_0^t\int_0^{t'} h(t'')dt''dt'\right)\right) + \int_0^t h(t')dt'$$

then by the above conversions, we get

$$u^r(t,r)=\frac{e(t)}{r}+\frac{1}{2r}\left[g\left(G^{-1}\left(r^2-\int_0^t\int_0^{t'}h(t'')dt''dt'\right)\right)\right].$$

No matter what ε is, we both have exact solution u^r. Next, we use the solutions to solve u^θ and p. It is obvious that $u^\theta=\frac{a}{r}$ with $a\in\mathbb{R}$ is a solution of (2.4) no matter how u^r is complex. In addition, we study the relatively difficult solution for u^θ. Let $u^r(t,r)=\frac{\varphi(t)}{r}+\frac{\psi(t)}{2}r$, then using characteristics' method, we get

$$u^\theta(t,r)=\tilde{u}^\theta(t,r_0)=\sigma(r_0)\exp\left[-\int_0^t\left(\frac{\varphi(t')}{r^2(t')}+\frac{\psi(t')}{2}\right)dt'\right] \tag{2.7}$$

with

$$\frac{1}{2}r^2(t)=r_0^2\frac{1}{2}\exp\left[\int_0^t\psi(t')dt'\right]+\exp\left[\int_0^t\psi(t')dt'\right]\int_0^t\varphi(t')\exp\left[-\int\psi(t')dt'\right]dt'$$

and $\psi(t)$ satisfies the ODE

$$\psi'=\varepsilon\psi^2+f(t)$$

Warning. It is careful to get the expression of $u^\theta(t,r)$. We must first deal with integration, than replace the r_0 by $r_0(t,r)$.

Therefore, applying the above results, we have the following consequence.

Theorem 2.1. Let $\alpha,\beta,\gamma,\delta$ be constants, $r=\sqrt{x^2+y^2}$, $k(t)=\gamma\exp(-\int\pi(t)dt)$, and the function $\pi(t)$ satisfies the ODE

$$\pi'=\pi^2+g(t,\pi),\ g(t,0)=0.$$

Then, three dimension incompressible Euler equations (1.4) has a class of exact solutions

$$\tilde{u}(t,x,y,z)=\left[\frac{x}{\sqrt{x^2+y^2}}u^r-\frac{\delta y}{x^2+y^2},\ \frac{y}{\sqrt{x^2+y^2}}u^r+\frac{\delta x}{x^2+y^2},\ -\left(\frac{1}{r}u^r+u^r_r\right)z\right] \tag{2.8}$$

$$p=-\frac{\delta^2}{2r^2}-\int u^r_t\,dr-\frac{1}{2}(u^r)^2+\frac{1}{2}z^2f(t)$$

where

$$u^r(t,r)=\frac{e(t)}{r}+\frac{r\pi(t)}{2}+\frac{\alpha}{2r}\exp\left[\int\left(3\pi(t)-\theta(t)k(t)\right)dt+r^2k(t)\right]$$
$$+\frac{\beta}{2r}\exp\left[\int\left(3\pi(t)+\theta(t)k(t)\right)dt-r^2k(t)\right]$$
$$f(t)=4\alpha\beta\gamma^2\exp\left[4\int\pi(t)dt\right]+g(t,\pi)$$

or

$$u^r(t,r)=\frac{e(t)}{r}+\frac{\sqrt{2}\gamma}{2r}\sin\left[\alpha r^2-\alpha\int\theta(t)dt+\beta+\frac{\pi}{4}\right],\ f(t)=-2\alpha^2\gamma^2$$

for any functions $e(t)$ and $\theta(t)$. In addition, it also has other kinds of exact solutions

$$\tilde{u}(t,x,y,z)=\left[\frac{x\varphi(t)}{x^2+y^2}+\frac{x\psi(t)}{2}-\frac{y}{\sqrt{x^2+y^2}}u^\theta,\ \frac{y\varphi(t)}{x^2+y^2}+\frac{y\psi(t)}{2}+\frac{x}{\sqrt{x^2+y^2}}u^\theta,\ -z\psi(t)\right] \tag{2.9}$$

$$\bar{p} = \int \frac{1}{r}(u^\theta)^2 dr - \varphi'(t)\ln r - \frac{1}{4}\psi'(t) r^2 - \frac{1}{2}\left(\frac{\varphi(t)}{r} + \frac{\psi(t)}{2} r\right)^2 + \frac{1}{2}(\psi' - \psi^2)(\ldots) \quad (2.10)$$

where u^θ satisfies (2.7), for any functions $\varphi(t)$ and $\psi(t)$.

Remark 2.2. About the exact solution of (1.4), there are many works,likely [9,10,11,12]. In this paper, we have a great improvement than [10]. In [10], the constructed solutions are

$$u^r = \frac{1}{2\alpha r}\left(\frac{1-e^{-\alpha r^2}}{t_0-t}\right), \quad u^\theta = 0, \quad u^z = -\frac{\alpha z}{t_0 - t}$$

with $\alpha \geq 0$, or

$$u^r = \frac{1}{2(1+\beta t)}\left(-\beta r + \frac{\delta(1-e^{-\alpha(1+\beta t)r^2})}{\alpha r(1+\beta t)[1-(\delta-\beta)t]}\right), \quad u^\theta = 0, \quad u^z = \frac{1}{1+\beta t}\left(\beta - \frac{\delta e^{-\alpha(1+\beta t)r^2}}{1-(\delta-\beta)t}\right)$$

with $\beta \geq 0$. In our first solutions, we can get the above solutions by choosing suitable parametric functions and constants. Choosing suitable parametric functions, we can get different solutions. But the energy is not finite.

Under the case two dimension compressible Euler equations, we let $f(t)=0$ and $\varepsilon=-1$, then

$$\psi(t) = \frac{1}{t+\beta}$$

Hence, using the above analyse, we also have the followingresult.

Theorem 2.3. Two dimension compressible Euler equations (1.2) has a sires of exact solutions

$$\bar{u}(t,x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}u^r - \frac{\alpha y}{x^2+y^2}, \frac{y}{\sqrt{x^2+y^2}}u^r + \frac{\alpha x}{x^2+y^2}\right),$$

$$p(t,x,y) = -\frac{\alpha^2}{2r^2} + \int \frac{u^r}{r} dr - \frac{1}{2}(u^r)^2 \quad (2.11)$$

with

$$u^r(t,r) = \frac{e(t)}{r} + \frac{1}{2r} g\left(G^{-1}\left(r^2 - \int_0^t \int_0^{t'} h(t'')dt''dt'\right)\right).$$

What is more, we also have other terms of solutions

$$\bar{u}(t,x,y) = \left(\frac{x\varphi(t)}{x^2+y^2} + \frac{x}{2(t+\beta)} - \frac{y}{\sqrt{x^2+y^2}}u^\theta, \frac{y\varphi(t)}{x^2+y^2} + \frac{y}{2(t+\beta)} + \frac{x}{\sqrt{x^2+y^2}}u^\theta\right), \quad (2.12)$$

$$\bar{p} = \int \frac{1}{r}(u^\theta)^2 dr - \varphi'(t)\ln r + \frac{r^2}{4(t+\beta)^2} - \frac{1}{2}\left(\frac{\varphi(t)}{r} + \frac{r}{2(t+\beta)}\right)^2 \quad (2.13)$$

with that u^θ meets (2.7) and $\psi(t) = \frac{1}{t+\beta}$.

3 The exact solutions for $n = 3$

In this section, we mainly consider the compressible Euler equations with $n = 3$. And, we give two class of especial solutions using the system (1.4).

3.1 The first class solutions

Let

$$u^r = u^r_1(r,t),\ u^\theta = u^\theta_1(r,t),\ u^z = u^z_1(r,t),\ p = p_1(r,t) \quad (3.1)$$

then we obtain the one-parameter model

$$\begin{cases} (\frac{1}{r}u^r_1 + u^r_{1r})_t + u^r_1(\frac{1}{r}u^r_1 + u^r_{1r})_r + (\frac{1}{r}u^r_1 + u^r_{1r})^2 = 0 \\ u^r_{1t} + u^r_1 u^r_{1r} - \frac{1}{r}(u^\theta_1)^2 + p_{1r} = 0 \\ u^\theta_{1t} + u^r_1 u^\theta_{1r} + \frac{1}{r}u^r_1 u^\theta_1 = 0 \\ u^z_{1t} + u^r_1 u^z_{1r} = 0 \end{cases} \quad (3.2)$$

Write

$$\eta = \frac{1}{r}u^r_1 + u^r_{1r}$$

Then, by ODE theorem we have

$$u^r_1 = \frac{e(t)}{r} + \frac{1}{r}\int r\eta dr$$

Applying the expression and (3.2), we get

$$\eta_t + (\frac{e(t)}{r} + \frac{1}{r}\int r\eta dr)\eta_r + \eta^2 = 0$$

Let

$$\eta(t,r) = w(t,z),\ z = r^2$$

we obtain

$$w_t + (2e(t) + \int wdz)w_z + w^2 = 0$$

Writing

$$\xi = 2e(t) + \int wdz,$$

then, in the end, we think about the problem

$$\xi_{zt} + \xi\xi_{zz} + \xi_z^2 = 0$$

By this equation, we get

$$\begin{cases} \xi_t + \xi\xi_z = h(t) \\ \xi(0,z) = g(z) \end{cases} \quad (3.3)$$

We use characteristics of the method to build solution. Hence, we have

$$\xi(t,z) = \tilde{\xi}(t,z_0) = g(z_0) + \int_0^t h(t')dt'$$

where z_0 meets

$$z = z_0 + tg(z_0) + \int_0^t \int_0^{t'} h(t'')dt''dt'$$

Choosing suitable $g(z_0)$ so that $G(z_0) = z_0 + tg(z_0) \in C^1$, then we have

$$\xi(t,z) = g\left(G^{-1}\left(z - \int_0^t\int_0^{t'} h(t'')dt''dt'\right) + \int_0^t h(t')dt'\right)$$

then by the above conversions, we get

$$u_1^r(t,r) = \frac{e(t)}{r} + \frac{1}{2r} g\left(G^{-1}\left(r^2 - \int_0^t\int_0^{t'} h(t'')dt''dt'\right)\right)$$

Choosing suitable $h(t)$, $g(z_0)$, we gain the solution u^r. It is obvious that for any constant β, $u_1^z = \beta$ is a solution of the last equation in (3.2) and $u_1^\theta = \frac{\alpha}{r}$ is also a solution of the third equation in (3.2), whatever u_1^r is. Moreover, we can check the claim that

$$u_1^r = \frac{e(t)}{r} + \frac{r}{2(t+\beta)}$$

satisfies the first equation in (3.2). Due to the characteristics' the method, we obtain the solution u_1^θ and u_1^z. For u^z, we have

$$u_1^z = \upsilon\left(\sqrt{\frac{\beta}{t+\beta}}\left(r^2 - 2\beta\int_0^t \frac{c(t')}{t'+\beta}dt'\right)\right)$$

As for u_1^θ, using the third equation and against the characteristics of the method, we get the result that the solution of the system

$$\begin{cases} u_{1t}^\theta + u_1^r u_{1r}^\theta + \frac{u_1^r u_1^\theta}{r} = 0 \\ u_1^\theta(0,r) = \psi(r) \end{cases}$$

has the solutions

$$u_1^\theta(t,r) = \tilde{u}_1^\theta(t,r_0) = \psi(r_0)\exp\left(-\int_0^t\left(\frac{c(t')}{r^2(t')} + \frac{1}{2(t'+\beta)}\right)dt'\right) \qquad (3.4)$$

with

$$\frac{1}{2}r^2(t) = \frac{t+\beta}{2\beta}r_0^2 + (t+\beta)\int_0^t \frac{c(t')}{t'+\beta}dt'$$

According to these works, we get the following results.

Theorem 3.1. Let α, β be constants meeting $\alpha \neq 0$, $\beta \neq 0$. Then the three dimension compressible Euler equations (1.2) has a class of exact solutions

$$\bar{u}(t,x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}}u_1^r - \frac{\alpha y}{x^2+y^2}, \frac{y}{\sqrt{x^2+y^2}}u^r + \frac{\alpha x}{x^2+y^2}, \beta\right), \qquad (3.5)$$

$$\bar{p}(t,x,y,z) = \int -u_{1t}^r dr - \frac{\alpha^2}{2r^2} - \frac{1}{2}(u_1^r)^2,\ \rho(t,x,y,z) = \frac{1}{r}u_1^r + u_{1r}^r$$

with

$$u_1^r(t,r) = \frac{e(t)}{r} + \frac{1}{2r} g\left(G^{-1}\left(r^2 - \int_0^t\int_0^{t'} h(t'')dt''dt'\right)\right)$$

Here, the function $G(s) = tg(s) + s$ is any invertible function. Moreover, it also has other exact solutions

$$\bar{u}(t,x,y,z) = \left(\frac{xc(t)}{x^2+y^2} + \frac{x}{2(t+\beta)} - \frac{y}{\sqrt{x^2+y^2}}u_1^\theta,\ \frac{yc(t)}{x^2+y^2} + \frac{y}{2(t+\beta)} + \frac{x}{\sqrt{x^2+y^2}}u_1^\theta,\ u_1^z \right) \quad (3.6)$$

$$\bar{p}(t,x,y,z) = -c'(t)\ln r + \frac{c^2}{2r^2} - \frac{r^2}{4(t+\beta)^2} - \frac{r^2}{2}\left(\frac{1}{t+\beta}\right)' + \int \frac{1}{r}(u_1^\theta)^2 dr,\ \rho(t,x,y,z) = \frac{1}{t+\beta} \quad (3.7)$$

where, $u_1^r = \frac{c(t)}{r} + \frac{r}{2(t+\beta)}$, and u_1^θ satisfies (3.4).

3.2 The second kind solutions

We consider the type solution

$$u^r = u_2^r(t, r),\ u^\theta = u_2^\theta(t, r),\ u^z = u_2^z(t, z),\ p = q_1(t, r) + q_2(t, z)$$

Then, we have the equations

$$\begin{cases} \left(\frac{1}{r}u_2^r + u_{2r}^r\right)_t + u_2^r\left(\frac{1}{r}u_2^r + u_{2r}^r\right)_r + u_2^z u_{2zz}^z + \left(\frac{1}{r}u_2^r + u_{2r}^r\right)^2 + 2\left(\frac{1}{r}u_2^r + u_{2r}^r\right)u_{2z}^z + (u_{2z}^z)^2 = 0 \\ u_{2t}^r + u_2^r u_{2r}^r - \frac{1}{r}(u_2^\theta)^2 + q_{1r} = 0 \\ u_{2t}^\theta + u_2^r u_{2r}^\theta + \frac{1}{r}u_2^r u_2^\theta = 0 \\ u_{2t}^z + u_2^z u_{2z}^z + q_{2z} = 0 \end{cases} \quad (3.8)$$

Let

$$\frac{1}{r}u_2^r + u_{2r}^r = \varphi(t)$$

then, we have

$$u_2^r = \frac{c(t)}{r} + \frac{1}{2}r\varphi(t) \quad (3.9)$$

and the reduced system

$$\begin{cases} \varphi' + \varphi^2 = 0 \\ u_{2zt}^z + u_2^z u_{2zz}^z + 2u_{2z}^z\varphi(t) + (u_{2z}^z)^2 = 0 \\ u_{2t}^r + u_2^r u_{2r}^r - \frac{1}{r}(u_2^\theta)^2 + q_{1r} = 0 \\ u_{2t}^\theta + u_2^r u_{2r}^\theta + \frac{1}{r}u_2^r u_2^\theta = 0 \\ u_{2t}^z + u_2^z u_{2z}^z + q_{2z} = 0 \end{cases} \quad (3.10)$$

By the first equation in (3.10), we get

$$\varphi(t) = \frac{1}{t+\beta},\ u_2^r = \frac{c(t)}{r} + \frac{r}{2(t+\beta)} \quad (3.11)$$

Using the second equation, we know that

$$\partial_z\left(u_{2t}^z + u_2^z u_{2z}^z + 2u_2^z\varphi(t)\right) = u_{2zt}^z + u_2^z u_{2zz}^z + 2u_{2z}^z\varphi(t) + (u_{2z}^z)^2 = 0$$

Combining with last equation, we have

$$u^z_{2t} + u^z_2 u^z_{2z} = -2u^z_2\varphi(t) + h(t)$$

with

$$q_{2z} = -2u^z_2\varphi(t) - h(t)$$

Employing the characteristics of the method, we know

$$u^z_2(t,z) = \tilde{u}^z_2(t, z_0) = \frac{\beta^2\pi(z_0)}{(t+\beta)^2} + \frac{1}{(t+\beta)^2}\int_0^t (t'+\beta)^2 h(t')dt'$$

Here z_0 satisfies

$$z = z_0 - \frac{\beta^2\pi(z_0)}{2(t+\beta)} + \int_0^t \frac{1}{(t'+\beta)^2}\int_0^{t'} (t''+\beta)^2 h(t'')dt''dt'$$

Choosing suitable function $\pi(z_0)$ is so that the function

$$\Pi(z_0) := z_0 - \frac{\beta^2\pi(z_0)}{2(t+\beta)}$$

belongs to C^1. Thus, we have

$$u^z_2 = \frac{\beta^2}{(t+\beta)^2}\pi\left(\Pi^{-1}\left(z - \int_0^t \frac{1}{(t'+\beta)^2}\int_0^{t'}(t''+\beta)^2 h(t'')dt''dt'\right)\right) + \frac{1}{(t+\beta)^2}\int_0^t (t'+\beta)^2 h(t')dt'$$

At the same time, we also get

$$q_2 = \frac{2}{t+\beta}\int u^z_2 dz - zh(t)$$

Transacting u^θ_2 is same as dealing with u^z_2. Thus, we have that

$$u^\theta_2(t,r) = \tilde{u}^\theta_2(t, r_0) = \varphi(r_0)\exp\left(-\int_0^t \left(\frac{c(t')}{r^2(t')} + \frac{1}{2(t'+\beta)}\right)dt'\right) \quad (3.12)$$

with

$$\frac{1}{2}r^2(t) = \frac{t+\beta}{\beta}r_0^2 + (t+\beta)\int_0^t \frac{c(t')}{t'+\beta}dt'$$

Now, due to the expression of u^θ_2 and u^z_2 and the third equation, the pressure q_1 is solved by

$$q_1 = \int \frac{1}{r}(u^\theta_2)^2 dr - \int u^z_2 dr - (u^z_2)^2$$

Therefore, we have the following consequence.

Theorem 3.2. Let $\beta \in \mathbb{R}$ and $r = \sqrt{x^2+y^2}$. And assume that the functions $\pi(s)$ satisfies the restrictions

$$\Pi(s) = s - \frac{\beta^2\pi(s)}{2(t+\beta)} \in C^1$$

Then, the three dimension compressible Euler equations (1.2) has a class of exact solutions

$$\bar{u}(t,x,y,z) = \left(\frac{xc(t)}{x^2+y^2} + \frac{x}{2(t+\beta)} - \frac{y}{\sqrt{x^2+y^2}}u^\theta, \frac{yc(t)}{x^2+y^2} + \frac{y}{2(t+\beta)} + \frac{x}{\sqrt{x^2+y^2}}u^\theta, u^z_2\right) \quad (3.13)$$

$$p(t,x,y,z) = \int \frac{1}{r}(u_2^{\beta})^2 dr - c(t) \ln r + \frac{r^2}{4(t+\beta)^2} - \frac{1}{2}\left(\frac{c(t)}{r} + \frac{r}{2(t+\beta)}\right)^2 + \int \frac{2}{t+\beta} u_2^z dz - zh(t),$$

$$\rho(t,x,y,z) = \frac{1}{t+\beta} + u_{2z}^z$$

Here,

$$u_2^z = \frac{\beta^2}{(t+\beta)^2}\pi(\Pi^{-1}\left(z - \int_0^t \frac{1}{(t'+\beta)^2}\int_0^{t'} (t''+\beta)^2 h(t'')dt''\, dt'\right) + \frac{1}{(t+\beta)^2}\int_0^t (t'+\beta)^2 h(t')dt'$$

and u_2^θ meets (3.12).

Remark 3.2.2. The two or three dimension compressible Euler equations's exact solutions in this paper, depend on the first order equation:

$$\begin{cases} u_t + uu_x = f \\ u(0,x) = \omega(x) \end{cases} \tag{3.14}$$

If $\omega(x)$ is Riemann's data, then the above equation has shock wave. Therefore, the two or three dimension compressible Euler equations has shock wave.

4 Conclusions

In this work, we utilize the system (1.3) to build up some exact solutions of the 2-dimensional and 3-dimensional compressible Euler equations. At the same time, we give some exact solutions for 3-dimensional incompressible Euler equations. However, the constructed exact solutions of incompressible system are infinite energetic, and simultaneously the blow-up solutions are also obtained via choosing certain proper variable functions.

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