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Solution of Euler's Differential Equation, and AC-Laplace Transform of Inverse Power Functions and Their Pseudofunctions, in Nonstandard Analysis

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Abstract

It is shown that the index law of the Riemann-Liouville fractional derivative is recovered when nonstandard analysis is applied, and then the solutions of Euler's differential equation are obtained in nonstandard analysis, where infinitesimal number ϵ appears. They are given in the form, from which the solutions in distribution theory are obtained. In the derivation, the AC-Laplace transforms of functions t^ν and $t^\nu(\log_e t)^m$ for complex number ν and positive integer m, are used. By using these formulas, the AC-Laplace transforms of functions $t^{-n+\epsilon}$ and $t^{-n+\epsilon}(\log_e t)^m$ for positive integers n and m, and their pseudofunctions are obtained with the aid of nonstandard analysis.

Keywords: Riemann-Liouville fractional derivative; Euler's differential equation; Laplace transform; AC-Laplace transform; nonstandard analysis; distribution theory; pseudofunction

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1 Introduction

In [1, 2], discussions are made of the solution of linear differential equations with polynomial coefficients, where the solutions are expressed by a linear combination of functions $g_\nu(t)$ defined by

$$g_\nu(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1} H(t), \quad \nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}, \tag{1}$$

where $\Gamma(\nu)$ is the gamma function, $t \in \mathbb{R}$, and $H(t)$ is Heaviside's step function which is equal to 0 if $t \le 0$ and to 1 if $t > 0$. In accordance with (1), we also use $g_\nu(t) = 0$ for $\nu \in \mathbb{Z}_{<1}$.

We use notations $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, which are the sets of all integers, all real numbers and all complex numbers, respectively, and also $\mathbb{Z}_{>a} = \{n \in \mathbb{Z}|n > a\}$, $\mathbb{Z}_{<a} = \{n \in \mathbb{Z}|n < a\}$ for $a \in \mathbb{Z}$, $\mathbb{R}_{>b} = \{x \in \mathbb{R}|x > b\}$ for $b \in \mathbb{R}$, and $\mathbb{C}_+ = \{z \in \mathbb{C}|\text{Re } z > 0\}$.

The Riemann-Liouville fractional derivative ${}_0D_R^\mu$ for $\mu \in \mathbb{C}$ is so defined that

$${}_0D_R^\mu g_\nu(t) = \begin{cases} g_{\nu-\mu}(t), & \nu-\mu \in \mathbb{C}\backslash\mathbb{Z}_{<1}, \\ g_{-n}(t) = 0, & \nu-\mu = -n \in \mathbb{Z}_{<1}. \end{cases} \tag{2}$$

This means that if $\nu - \mu_1 = -n \in \mathbb{Z}_{<1}$, $\nu - \mu_2 \notin \mathbb{Z}_{<1}$ and $\nu - \mu_1 - \mu_2 \notin \mathbb{Z}_{<1}$,

$$\begin{aligned} {}_0D_R^{\mu_2}\,{}_0D_R^{\mu_1} g_\nu(t) &= {}_0D_R^{\mu_2} g_{-n}(t) = 0 \ne {}_0D_R^{\mu_1}\,{}_0D_R^{\mu_2} g_\nu(t) = {}_0D_R^{\mu_1} g_{\nu-\mu_2}(t) = g_{\nu-\mu_2-\mu_1}(t) \\ &= {}_0D_R^{\mu_1+\mu_2} g_\nu(t) \ne 0. \end{aligned} \tag{3}$$

For example, if $\nu = \frac{3}{2}$, $\mu_1 = \frac{3}{2}$ and $\mu_2 = \frac{1}{2}$, we have

$$\begin{aligned} {}_0D_R^{1/2}\,{}_0D_R^{3/2} g_{3/2}(t) &= {}_0D_R^{1/2} g_0(t) = 0 \ne {}_0D_R^{3/2}\,{}_0D_R^{1/2} g_{3/2}(t) = {}_0D_R^{3/2} g_1(t) = g_{-1/2}(t) \\ &= {}_0D_R^{3/2+1/2} g_{3/2}(t) \ne 0. \end{aligned} \tag{4}$$

This example is given in [3, p. 108].

We now study the problem with the aid of nonstandard analysis [4]. We then use infinitesimal numbers belonging to $\mathbb{R}$ and $\mathbb{C}$; see [4]. We denote them by $\mathbb{R}^0$ and $\mathbb{C}^0$, respectively. We also use $\mathbb{R}^0_{\ne 0} = \{x \in \mathbb{R}^0|x \ne 0\}$, $\mathbb{C}^0_{\ne 0} = \{z \in \mathbb{C}^0|z \ne 0\}$, $\mathbb{R}^0_{>0} = \{x \in \mathbb{R}^0|x > 0\}$ and $\mathbb{C}^0_+ = \{z \in \mathbb{C}^0|\text{Re } z > 0\}$.

We note here that if $\epsilon \in \mathbb{R}^0_{>0}$ and $x \in \mathbb{R}_{>0}\backslash\mathbb{R}^0_{>0}$, ϵ is so small that $0 < \epsilon < x$.

When $\nu \in \mathbb{Z}_{<1}$, we use $g_{\nu+\epsilon}(t)$ in place of $g_\nu(t)$, where $\epsilon \in \mathbb{C}^0_{\ne 0}$. In place of (3), we then have

$$\begin{aligned} {}_0D_R^{\mu_2}\,{}_0D_R^{\mu_1} g_{\nu+\epsilon}(t) &= {}_0D_R^{\mu_2} g_{-n+\epsilon}(t) = g_{-n+\epsilon-\mu_2}(t) = {}_0D_R^{\mu_1}\,{}_0D_R^{\mu_2} g_{\nu+\epsilon}(t) = {}_0D_R^{\mu_1} g_{\nu+\epsilon-\mu_2}(t) \\ &= {}_0D_R^{\mu_1+\mu_2} g_{\nu+\epsilon}(t) \ne 0. \end{aligned} \tag{5}$$

We see that the index law for the operator ${}_0D_R$ does not hold in (3), but it holds in (5).

The Laplace transform of $g_\nu(t)$ is denoted by $\mathcal{L}[g_\nu(t)]$ for $\nu \in \mathbb{C}_+$, which is given by

$$\mathcal{L}[g_\nu(t)] = \frac{1}{\Gamma(\nu)} \mathcal{L}[t^{\nu-1}] = \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} e^{-st} dt = s^{-\nu}, \quad \nu \in \mathbb{C}_+, \tag{6}$$

and the AC-Laplace transform, that is the analytic continuation of the Laplace transform, of $g_\nu(t)$ is denoted by $\mathcal{L}_H[g_\nu(t)]$ for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, which is given in [1] by

$$\mathcal{L}_H[g_\nu(t)] = s^{-\nu}, \quad \nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}. \tag{7}$$

When $\nu = -n \in \mathbb{Z}_{<1}$, we use

$$\mathcal{L}_H[g_{-n+\epsilon}(t)] = \frac{1}{\Gamma(-n+\epsilon)}\mathcal{L}_H[t^{-n-1+\epsilon}] = s^{n-\epsilon}, \tag{8}$$

where $\epsilon \in \mathbb{C}^0_{\neq 0}$.

By (6) and (7), a function is not given for which the AC-Laplace transform is s^n for $n \in \mathbb{Z}_{>-1}$, but (8) shows that the AC-Laplace transform of $g_{-n+\epsilon}(t)$ is $s^{n-\epsilon}$, where $\epsilon \in \mathbb{C}^0_{\neq 0}$. In Section 6, we discuss the AC-Laplace transforms of $t^{-n+\epsilon}$ and $t^{-n+\epsilon}(\log_e t)^m$ for $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{>0}$, by using Formula (8). In Section 5, we use $\mathcal{L}_H[g_{\nu+\epsilon}(t)]$ in place of $\mathcal{L}_H[g_\nu(t)]$.

Remark 1.1. It was mentioned in [5], that this observation justifies the proposal of Ghil and Kim given in [6], in which the inverse Laplace transform of C is given by t^{-1}, where C is a constant; although their choice $C = -1$ cannot be justified.

In Section 2, a short explanation of distribution theory is given. When we consider the Laplace transform of $g_\nu(t)$ for $\nu \in \mathbb{C}_+$ in distribution theory, we consider regular distribution $\tilde{g}_\nu(t) = g_\nu(t)\tilde{H}(t)$, which corresponds to function $g_\nu(t)H(t)$. Then we have

$$\langle \tilde{g}_\nu(t), e^{-st}\rangle = \langle g_\nu(t)\tilde{H}(t), e^{-st}\rangle = \frac{1}{\Gamma(\nu)}\int_0^\infty t^{\nu-1}e^{-st}dt = \mathcal{L}[g_\nu(t)] = \mathcal{L}_H[g_\nu(t)] = s^{-\nu}. \tag{9}$$

When $\nu \in \mathbb{C}_+$, $n \in \mathbb{Z}_{>0}$, $\nu - n \notin \mathbb{C}_+$ and $\nu - n \notin \mathbb{Z}_{<1}$, we adopt $\tilde{g}_{\nu-n}(t) = \tilde{g}^{(n)}_\nu(t) := D^n\tilde{g}_\nu(t)$, and then we have

$$\begin{aligned}\langle \tilde{g}_{\nu-n}(t), e^{-st}\rangle &= \langle \tilde{g}^{(n)}_\nu(t), e^{-st}\rangle = \langle D^n[g_\nu(t)\tilde{H}(t)], e^{-st}\rangle = s^n\langle g_\nu(t)\tilde{H}(t), e^{-st}\rangle \\ &= s^{n-\nu} = \mathcal{L}_H[g_{\nu-n}(t)] = \mathcal{L}_H[\frac{d^n}{dt^n}g_\nu(t)].\end{aligned} \tag{10}$$

Remark 1.2. When $\nu - n \in \mathbb{Z}_{<1}$, we use Equation (10) with ν replaced by $\nu + \epsilon$, where $\epsilon \in \mathbb{C}^0_{\neq 0}$, in accordance with Equation (8).

In solving an ordinary differential equation, when functions which are not differentiable on $\mathbb{R}$, distribution theory is used, where the differential operators D^n for $n \in \mathbb{Z}$ satisfy the index law. When we use the Riemann-Liouville differential operators ${}_0D^\mu_R$ for $\mu \in \mathbb{C}$, because they do not satisfy the index law, we depend on distribution theory. In distribution theory, distributions are used in place of functions. Since distributions are functionals, they are not so familiar as functions. In the present paper, we use nonstandard analysis, where infinitesimal numbers and infinite numbers are used in the space of $\mathbb{R}$. By including these numbers, the index law of the Riemann-Liouville differential operators ${}_0D^\mu_R$ is recovered, and we can solve the problems by using only functions, without invoking distribution theory.

In Section 5, we recall the therems on the solution of Euler's differential equation, which are given in [5]. We then show that the solutions of Euler's differential equation in nonstandard analysis can be given in the form, from which the solutions in distribution theory are obtained, with the aid of Remark 1.2. For the preparation, we give the Laplace transform of $t^{\nu-1}(\log_e t)^m$ for $\nu \in \mathbb{C}_+$ and $m \in \mathbb{Z}_{>0}$ in Section 3, and the AC-Laplace transform of $t^{-1+z}(\log_e t)^m$ for $m \in \mathbb{Z}_{>-1}$ and $z \in \mathbb{C}$ which satisfies $0 < |z| \ll 1$, in Section 4.

We know that the AC-Laplace transforms of functions t^{-n} and $t^{-n}(\log_e t)^m$ for $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{>0}$, do not exist. In distribution theory, the Laplace transforms of pseudofunctions Pf $t^{-n}H(t)$ and Pf $t^{-n}(\log_e t)^m \cdot H(t)$, are defined in their places [7, pp. 17, 57]. In Section 6, we show that the AC-Laplace transforms of functions $t^{-n+\epsilon}$ and $t^{-n+\epsilon}(\log_e t)^m$, and of pseudofunctions Pf $t^{-n+\epsilon}$ and Pf $t^{-n+\epsilon}(\log_e t)^m$, are obtained with the aid of nonstandard analysis, by using the formulas given in Sections 3 and 4.

2 Preliminaries on Distribution Theory

Distributions in the space $\mathcal{D}'$ are first introduced in [8, 9, 10, 7]. The distributions are either regular or nonregular. A regular distribution in $\mathcal{D}'$ corresponds to a function $f(t)$ which is locally integrable on $\mathbb{R}$. We denote the distribution by $\tilde{f}(t)$.

A distribution $\tilde{u} \in \mathcal{D}'$ is a functional, to which $\langle \tilde{u}, \phi \rangle \in \mathbb{C}$ is associated with every $\phi \in \mathcal{D}$, where $\mathcal{D}$, that is dual to $\mathcal{D}'$, is the space of testing functions, which are infinitely differentiable and have a compact support on $\mathbb{R}$.

If $\tilde{f} \in \mathcal{D}'$ is a regular distribution, we have

$$\langle \tilde{f}, \phi \rangle = \int_{-\infty}^{\infty} f(t)\phi(t)dt. \tag{11}$$

A distribution which is not regular is expressed by $D^k \tilde{f}(t)$ in terms of an operator D, $k \in \mathbb{Z}_{>0}$ and a regular distribution $\tilde{f}(t)$. In this case, $\langle D^k \tilde{f}, \phi \rangle = \langle \tilde{f}, D_W^k \phi \rangle$, where $D_W = -\frac{d}{dt}$. Because of this definition of D, we can confirm the following lemma.

Lemma 2.1. *Let $\tilde{f}$ and $\tilde{g}$ be regular distributions in $\mathcal{D}'$, which correspond to $f(t)$ and $\frac{d}{dt}f(t)$, respectively. Then $\tilde{g} = D\tilde{f}$.*

Proof. In this condition, we have

$$\langle D\tilde{f}, \phi \rangle = \langle \tilde{f}, D_W \phi \rangle = -\int_{-\infty}^{\infty} f(t)[\frac{d}{dt}\phi(t)]dt = \int_{-\infty}^{\infty} [\frac{d}{dt}f(t)]\phi(t)dt = \langle \tilde{g}, \phi \rangle. \tag{12}$$

■

When we discuss the Laplace transform, we consider distributions in the space $\mathcal{S}'_R$, which is a subspace of $\mathcal{D}'$. The space $\mathcal{S}_R$, which is dual to $\mathcal{S}'_R$, consists of functions which are infinitely differentiable on $\mathbb{R}$, and decay rapidly as $t \to \infty$. We denote the space of regular distributions in $\mathcal{S}'_R$ by $\mathcal{S}'_{R,reg}$. When $\tilde{f} \in \mathcal{S}'_{R,reg}$ corresponds to a function $f(t)$, $f(t)$ is locally integrable on $\mathbb{R}$ and increases slowly as $t \to \infty$. These conditions require that if $\phi \in \mathcal{S}_R$, $l \in \mathbb{Z}_{>-1}$ and $N \in \mathbb{Z}_{>0}$, $|\phi^{(l)}(t)|t^N \to 0$ as $t \to \infty$, if $f(t) \in \mathcal{S}'_{R,reg}$, there exists $M \in \mathbb{Z}_{>0}$ for which $|f(t)|t^{-M} \to 0$ as $t \to \infty$, and hence the product $f(t)\phi(t)$ tends to 0 as $t \to \infty$,

We note that the distribution $\tilde{H}(t)$ which corresponds to $H(t)$ belongs to $\mathcal{S}'_{R,reg}$, and $g_\nu(t)\tilde{H}(t)$ which appeared in (9) belongs to $\mathcal{S}'_{R,reg}$, so that it corresponds to $g_\nu(t)H(t)$, if $\nu \in \mathbb{C}_+$.

Lemma 2.2. *Let $f(t)\tilde{H}(t) \in \mathcal{S}'_{R,reg}$ and $(\frac{d}{dt}f(t))\tilde{H}(t) \in \mathcal{S}'_{R,reg}$. Then*

$$(\frac{d}{dt}f(t))\tilde{H}(t) = D[f(t)\tilde{H}(t)] - f(0)\delta(t). \tag{13}$$

Proof.

$$\langle (\frac{d}{dt}f(t))\tilde{H}(t), \phi(t) \rangle = \int_0^{\infty} (\frac{d}{dt}f(t))\phi(t)dt = f(t)\phi(t)|_{t=0}^{\infty} - \int_0^{\infty} f(t)\frac{d}{dt}\phi(t)dt$$
$$= \langle f(t)\tilde{H}(t), D_W\phi(t) \rangle - \langle f(0)\delta(t), \phi(t) \rangle = \langle D[f(t)\tilde{H}(t)] - f(0)\delta(t), \phi(t) \rangle, \tag{14}$$

for $\phi(t) \in \mathcal{S}_R$. ■

Corollary 2.1. *Let $f(t)\tilde{H}(t) \in \mathcal{S}'_{R,reg}$, $(\frac{d}{dt}f(t))\tilde{H}(t) \in \mathcal{S}'_{R,reg}$, and $f(0) = 0$. Then $(\frac{d}{dt}f(t))\tilde{H}(t) = D[f(t)\tilde{H}(t)]$.*

3 Laplace transforms of $(\log_e t)^m$ and $t^{\nu-1}(\log_e t)^m$ for $\nu \in \mathbb{C}_+$ and $m \in \mathbb{Z}_{>0}$

In this section, we put $\nu \in \mathbb{C}_+$, $z \in \mathbb{C}$ and $0 < |z| < \text{Re}\,\nu$, so that $\nu + z \in \mathbb{C}_+$, and use

$$t^{\nu+z-1} = t^{\nu-1}[1 + \sum_{m=1}^{\infty} \frac{1}{m!} z^m (\log_e t)^m]. \tag{15}$$

We then obtain

$$\begin{aligned} \frac{1}{\Gamma(\nu)} \mathcal{L}[t^{\nu-1+z}] &= \frac{1}{\Gamma(\nu)} \mathcal{L}[t^{\nu-1} + \sum_{m=1}^{\infty} \frac{1}{m!} z^m t^{\nu-1} (\log_e t)^m] \\ &= \frac{1}{s^\nu} + \frac{1}{\Gamma(\nu)} (z \cdot \mathcal{L}[t^{\nu-1} \log_e t] + \frac{1}{2} z^2 \cdot \mathcal{L}[t^{\nu-1} (\log_e t)^2]) + O(z^3). \end{aligned} \tag{16}$$

Equation (6) gives

$$\begin{aligned} \frac{1}{\Gamma(\nu)} \mathcal{L}[t^{\nu-1+z}] &= \frac{\Gamma(\nu+z)}{\Gamma(\nu)} s^{-\nu-z} = \frac{1}{s^\nu} [1 + \sum_{k=1}^{\infty} \frac{1}{k!} z^k \frac{\Gamma^{(k)}(\nu)}{\Gamma(\nu)}][1 + \sum_{l=1}^{\infty} (-1)^l \frac{1}{l!} z^l (\log_e s)^l] \\ &= \frac{1}{s^\nu} [1 + z(\frac{\Gamma'(\nu)}{\Gamma(\nu)} - \log_e s) + \frac{1}{2} z^2 (\frac{\Gamma''(\nu)}{\Gamma(\nu)} - 2 \frac{\Gamma'(\nu)}{\Gamma(\nu)} \log_e s + (\log_e s)^2) \\ &\quad + O(z^3)]. \end{aligned} \tag{17}$$

By equating the terms of $O(z)$, $O(z^2)$ and $O(z^m)$ in the righthand sides of Equations (16) and (17), we obtain

$$\frac{1}{\Gamma(\nu)} \mathcal{L}[t^{\nu-1} \log_e t] = \frac{1}{s^\nu} (\frac{\Gamma'(\nu)}{\Gamma(\nu)} - \log_e s), \tag{18}$$

$$\frac{1}{\Gamma(\nu)} \mathcal{L}[t^{\nu-1} (\log_e t)^2] = \frac{1}{s^\nu} (\frac{\Gamma''(\nu)}{\Gamma(\nu)} - 2 \frac{\Gamma'(\nu)}{\Gamma(\nu)} \log_e s + (\log_e s)^2), \tag{19}$$

$$\frac{1}{\Gamma(\nu)} \mathcal{L}[t^{\nu-1} (\log_e t)^m] = \frac{1}{s^\nu} \sum_{l=0}^{m} \frac{m!}{(m-l)!l!} \frac{\Gamma^{(m-l)}(\nu)}{\Gamma(\nu)} (-1)^l (\log_e s)^l, \quad m \in \mathbb{Z}_{>0}. \tag{20}$$

When $\nu = 1$, by using

$$\begin{aligned} \psi(z) &= \frac{d}{dz} \log_e \Gamma(z), \quad \psi(1) = \Gamma'(1) = -\gamma = -0.5772\cdots, \\ \psi'(z) &= \frac{\Gamma''(z)}{\Gamma(z)} - \psi(z)^2, \quad \psi'(1) = \zeta(2) = \frac{\pi^2}{6}, \end{aligned} \tag{21}$$

[11, 6.3.1, 6.3.2, 6.4.2 and 23.2.24], in (18)∼(20), we obtain

$$\mathcal{L}[\log_e t] = \frac{1}{s} (-\gamma - \log_e s), \tag{22}$$

$$\mathcal{L}[(\log_e t)^2] = \frac{1}{s} (\gamma^2 + \frac{\pi^2}{6} + 2\gamma \log_e s + (\log_e s)^2), \tag{23}$$

$$\mathcal{L}[(\log_e t)^m] = \frac{1}{s} \sum_{l=0}^{m} \frac{m!}{(m-l)!l!} \Gamma^{(m-l)}(1) (-1)^l (\log_e s)^l, \quad m \in \mathbb{Z}_{>0}. \tag{24}$$

Lemma 3.1. *Let $\nu \in \mathbb{C}_+$, $m \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{>0}$. Then*

$$\langle t^{\nu-1}(\log_e t)^m \tilde{H}(t), e^{-st}\rangle = \mathcal{L}[t^{\nu-1}(\log_e t)^m], \tag{25}$$

$$\langle D^n[t^{\nu-1}(\log_e t)^m \tilde{H}(t)], e^{-st}\rangle = s^n\langle t^{\nu-1}(\log_e t)^m \tilde{H}(t), e^{-st}\rangle = \mathcal{L}_H[\frac{d^n}{dt^n}[t^{\nu-1}(\log_e t)^m]]. \tag{26}$$

When $\nu - 1 - n \in \mathbb{Z}_{<1}$, we use (26) with ν replaced by $\nu + \epsilon$, following Remark 1.2.

Proof. When $\nu + z \in \mathbb{C}_+$, Equation (9) shows that $\langle t^{\nu+z-1}\tilde{H}(t), e^{-st}\rangle = \mathcal{L}[t^{\nu+z-1}]$. By using (15), in the both sides of this equation, we obtain (25), if $\nu+z \in \mathbb{C}_+$ and $\nu \in \mathbb{C}_+$. When $\nu+z \in \mathbb{C}_+$ and $n \in \mathbb{Z}_{>0}$, Equation (10) shows that $\langle D^n[t^{\nu+z-1}\tilde{H}(t)], e^{-st}\rangle = s^n\langle t^{\nu+z-1}\tilde{H}(t), e^{-st}\rangle = \mathcal{L}_H[\frac{d^n}{dt^n}t^{\nu+z-1}]$. By using (15) in the both sides of this equation, we obtain (26), if $\nu + z \in \mathbb{C}_+$ and $\nu \in \mathbb{C}_+$. ■

4 Laplace Transform of t^{-1+z} and $t^{-1+z}(\log_e t)^m$ for $m \in \mathbb{Z}_{>0}$ and $z \in \mathbb{C}$ Satisfying $0 < |z| < 1$

In this section, $z \in \mathbb{C}$, $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{C}$, which satisfy $0 < |z_2| < 1$ and $0 < |z_1| < 1$.

When $n \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned}\mathcal{L}_H[t^{-n+z_1+z_1z_2}] &= \mathcal{L}_H[t^{-n+z_1}(1 + \sum_{m=1}^{\infty}\frac{1}{m!}z_1^m z_2^m(\log_e t)^m)] \\ &= \mathcal{L}_H[t^{-n+z_1}] + \sum_{m=1}^{\infty}\frac{1}{m!}z_1^m z_2^m \mathcal{L}_H[t^{-n+z_1}(\log_e t)^m].\end{aligned} \tag{27}$$

By using (7) for $\nu = z$ and $\nu = z + 1$, and then (16) for $\nu = 1$, we obtain

$$\begin{aligned}\mathcal{L}_H[t^{-1+z}] &= \Gamma(z)s^{-z} = \frac{\Gamma(z+1)}{z}s^{-z} = \frac{s}{z}\mathcal{L}[t^z] = \frac{1}{z} + \sum_{l=1}^{\infty}\frac{s}{l!}z^{l-1}\mathcal{L}[(\log_e t)^l] \\ &= \frac{1}{z} + s\cdot\mathcal{L}[\log_e t] + \frac{1}{2}zs\cdot\mathcal{L}[(\log_e t)^2] + O(z^2).\end{aligned} \tag{28}$$

Lemma 4.1. *Let $m \in \mathbb{Z}_{>-1}$. Then*

$$\mathcal{L}_H[t^{-1+z_1}(\log_e t)^m] = \frac{(-1)^m m!}{z_1^{m+1}} + \frac{s}{m+1}\mathcal{L}[(\log_e t)^{m+1}] + O(z_1), \tag{29}$$

where $\mathcal{L}[(\log_e t)^{m+1}]$ is given in (24).

Proof. When $m > 0$, the lefthand side of (29) is the term of order z_2^m in (27) for $n = 1$, multiplied by $\frac{m!}{z_1^m}$. The righthand side is the sum of corresponding terms in the fifth member of (28) for $z = z_1(1+z_2)$, where the first two terms are due to the terms $\frac{1}{z} = \frac{1}{z_1(1+z_2)}$ and $\frac{s}{l!}z_1^{l-1}(1+z_2)^{l-1}\mathcal{L}[(\log_e t)^l]$ for $l = m+1$. When $m = 0$, (29) is due to (28) for $z = z_1$. ■

5 Solution of Euler's Differential Equation

In this section, we study the solution of the equation:

$$D_t^0 u(t) := t^n\frac{d^n}{dt^n}u(t) + \sum_{k=0}^{n-1}a_k\cdot t^k\frac{d^k}{dt^k}u(t) = 0, \quad t > 0, \tag{30}$$

where $n \in \mathbb{Z}_{>0}$, and a_k are constants, among which we adopt $a_n = 1$. This equation is called Euler's differential equation [12, Section 6.3], [13, Chapter II, Section 7].

We now present four theorems for the solution of Euler's equation, given in [5].

When D_t^0 given in (30) is operated to t^α for $\alpha \in \mathbb{C}$, we have

$$D_t^0 t^\alpha = A_0(\alpha) t^\alpha, \tag{31}$$

where

$$A_0(\alpha) := \sum_{k=0}^{n} a_k \cdot (\alpha)_k^-. \tag{32}$$

Then $A_0(\alpha)$ is a polynomial of degree n. Let $k_x \in \mathbb{Z}_{>0}$ be the total number of distinct roots of $A_0(\alpha) = 0$, which are α_k for $k \in \mathbb{Z}_{[0,k_x]}$. Then $A_0(\alpha)$ is expressed as

$$A_0(\alpha) = \prod_{k=1}^{k_x} (\alpha - \alpha_k)^{m_k}, \tag{33}$$

where $m_k \in \mathbb{Z}_{>0}$ for $k \in \mathbb{Z}_{[1,k_x]}$ satisfy $\sum_{k=1}^{k_x} m_k = n$. Now Equation (30) is expressed by

$$D_t^0 u(t) = \prod_{k=1}^{k_x} (t\frac{d}{dt} - \alpha_k)^{m_k} u(t) = 0, \quad t > 0. \tag{34}$$

Theorem 5.1. We have n solutions of Equation (30), which is expressed by Equation (34). They are classified into k_x series. In the kth series, if $m_k = 1$, we have one solution given by t^{α_k}, and if $m_k \geq 2$, we have m_k solutions given by

$$t^{\alpha_k},\ t^{\alpha_k} \log_e t,\ \cdots,\ t^{\alpha_k} (\log_e t)^{m_k - 1}. \tag{35}$$

We present the following theorems in the form taking account of nonstandard analysis. We use $\epsilon \in \mathbb{R}^0_{>0}$,

$$H_\epsilon(t) = t^\epsilon H(t), \tag{36}$$

in place of $H(t)$, distribution $\tilde{H}_\epsilon(t)$ which corresponds to $H_\epsilon(t)$, and $u_\nu(t)$ which is defined by

$$u_\nu(t) = \frac{t^\nu}{\Gamma(\nu + 1 + \epsilon)}, \tag{37}$$

for all $\nu \in \mathbb{C}$.

Remark 5.1. In the following, $u_{-1}(t)t^\epsilon$ appears often. We note that it is expressed as

$$u_{-1}(t)t^\epsilon = \frac{t^{\epsilon-1}}{\Gamma(\epsilon)} = \frac{\epsilon t^{\epsilon-1}}{\Gamma(\epsilon+1)} = \epsilon t^{\epsilon-1}(1 + O(\epsilon)). \tag{38}$$

We adopt the following equation which corresponds to (34), in distribution theory:

$$\prod_{k=1}^{k_x} (tD - \alpha_k - \epsilon)^{m_k} \tilde{u}(t) = 0. \tag{39}$$

We now present Theorem 4 in [5] in the form taking account of nonstandard analysis.

Theorem 5.2. Let the condition of Theorem 5.1 be satisfied, and $\epsilon \in \mathbb{R}^0_{>0}$. Then we have k_x series of solutions of Equation (39). The solutions in the kth series are given as follows.
(i) When Re $\alpha_k \geq -1$, if $m_k = 1$, $u_{\alpha_k}(t)\tilde{H}_\epsilon(t)$ is a solution, and if $m_k \geq 2$, we have m_k solutions given by

$$u_{\alpha_k}(t)\tilde{H}_\epsilon(t),\ u_{\alpha_k}(t)(\log_e t)^l \tilde{H}_\epsilon(t), \quad l \in \mathbb{Z}_{[1,m_k-1]}. \tag{40}$$

(ii) When Re $\alpha_k < -1$, we put $-\mu_k - 1 = \lfloor \text{Re}\ \alpha_k \rfloor$, which is the greatest integer which is not greater than Re α_k, and $\alpha_k = -\mu_k - 1 + \lambda_k$, so that $0 \leq \text{Re}\ \lambda_k < 1$, and then if $m_k = 1$, we have one solution given by $D^{\mu_k}[u_{-1+\lambda_k}(t)\tilde{H}_\epsilon(t)]$, and if $m_k \geq 2$, we have m_k solutions given by

$$D^{\mu_k}[u_{-1+\lambda_k}(t)\tilde{H}_\epsilon(t)],\ D^{\mu_k}[u_{-1+\lambda_k}(t)(\log_e t)^l \tilde{H}_\epsilon(t)], \quad l \in \mathbb{Z}_{[1,m_k-1]}, \tag{41}$$

so that $u_{-1+\lambda_k}(t)\tilde{H}_\epsilon(t)$ and $u_{-1+\lambda_k}(t)(\log_e t)^l \tilde{H}_\epsilon(t)$ for $l \in \mathbb{Z}_{1,m_k-1}$ are regular distributions.

In particular, when $\alpha_k = -1$, if $m_k = 1$, $u_{-1}(t)\tilde{H}_\epsilon(t)$ is a solution, and if $m_k \geq 2$, we have m_k solutions given by

$$u_{-1}(t)\tilde{H}_\epsilon(t),\ u_{-1}(t)(\log_e t)^l \tilde{H}_\epsilon(t), \quad l \in \mathbb{Z}_{[1,m_k-1]}, \tag{42}$$

and when $\alpha_k \in \mathbb{Z}_{<-1}$, we put $\mu_k = -\alpha_k - 1$, and then if $m_k = 1$, $D^{\mu_k}[u_{-1}(t)\tilde{H}_\epsilon(t)]$ is a solution, and if $m_k \geq 2$, we have m_k solutions given by

$$D^{\mu_k}[u_{-1}(t)\tilde{H}_\epsilon(t)],\ D^{\mu_k}[u_{-1}(t)(\log_e t)^l \tilde{H}_\epsilon(t)], \quad l \in \mathbb{Z}_{[1,m_k-1]}. \tag{43}$$

The following two theorems taking account of nonstandard analysis corresponds to Theorems 7 and 8 in [5].

We adopt the following equation which corresponds to (34) and (39):

$$\prod_{k=1}^{k_x} (t\frac{d}{dt} - \alpha_k - \epsilon)^{m_k} u(t) = 0, \quad t > 0. \tag{44}$$

Theorem 5.3. Let the condition in Theorem 5.1 be satisfied, and $\epsilon \in \mathbb{R}^0_{>0}$. Then we have n solutions of Equation (44), which are classified into k_x series. If $m_k = 1$, $u_{\alpha_k}(t)t^\epsilon$ is a solution in the kth series, and if $m_k \in \mathbb{Z}_{>1}$,

$$u_{\alpha_k}(t)t^\epsilon, \quad u_{\alpha_k}(t)t^\epsilon(\log_e t)^l, \quad l \in \mathbb{Z}_{[1,m_k-1]}. \tag{45}$$

are solutions in it.

In particular, when $\alpha_k = -\mu_k - 1 \in \mathbb{Z}_{<0}$, if $m_k = 1$,

$$u_{-\mu_k-1}(t)t^\epsilon = \epsilon(-1)^{\mu_k}\mu_k! t^{\epsilon-\mu_k-1}H(t) \tag{46}$$

is a solution in the kth series, and if $m_k \in \mathbb{Z}_{>1}$,

$$u_{-\mu_k-1}(t)t^\epsilon, \quad u_{-\mu_k-1}(t)t^\epsilon(\log_e t)^l, \quad l \in \mathbb{Z}_{[1,m_k-1]}, \tag{47}$$

are solutions in it.

Proof. If $\mu_k = n \in \mathbb{Z}_{>0}$, Equation (46) is obtained as

$$u_{-n-1}(t)t^\epsilon = \frac{d^n}{dt^n}\frac{t^{\epsilon-1}}{\Gamma(\epsilon)} = \frac{d^n}{dt^n}\frac{\epsilon t^{\epsilon-1}}{\Gamma(\epsilon+1)} = \frac{\epsilon(\epsilon-1)^-_n}{\Gamma(\epsilon+1)} t^{\epsilon-n-1} = \epsilon(-1)^n n! t^{\epsilon-n-1} + O(\epsilon^2), \tag{48}$$

where

$$(\epsilon-1)^-_n = (\epsilon-1)(\epsilon-2)\cdots(\epsilon-n). \tag{49}$$

The equation for $\mu_k = 0$ is given in Remark 5.1. ∎

Theorem 5.4. Let the condition in Theorem 5.1 be satisfied, and $\epsilon \in \mathbb{R}^0_{>0}$. Then the Laplace transform $\hat{u}(s)$ of a solution of Equation (44) satisfies

$$\prod_{k=1}^{k_x}(-\frac{d}{ds}s - \alpha_k - \epsilon)^{m_k}\hat{u}(s) = \prod_{k=1}^{k_x}(-s\frac{d}{ds} - 1 - \alpha_k - \epsilon)^{m_k}\hat{u}(s) = 0, \tag{50}$$

and we have n solutions of Equation (50), which are classified into k_x series. If $m_k = 1$,

$$\hat{u}_{\alpha_k}(s) = s^{-\alpha_k - 1 - \epsilon} \tag{51}$$

is a solution in the kth series, and if $m_k \in \mathbb{Z}_{>1}$,

$$\hat{u}_{\alpha_k}(s), \quad \hat{u}_{\alpha_k}(s)(\log_e s)^l, \quad l \in \mathbb{Z}_{[1,m_k-1]}, \tag{52}$$

are solutions in it. We obtain n solutions of Equation (44) by the inverse Laplace transform of the n solutions of Equation (50).

The solutions given in Theorem 5.3 take a form which is not convenient to be compared with the solutions given in Theorem 5.2. We construct the solutions in a different form by using the following lemma.

Lemma 5.1. *Let α be expressed as $\alpha = \lambda - \mu$ by $\mu \in \mathbb{Z}_{>-1}$ and $\lambda \in \mathbb{C}_+$, and $v(t)$ be a solution of*

$$(t\frac{d}{dt} - \lambda)^m v(t) = 0. \tag{53}$$

Then $u(t) = \frac{d^\mu}{dt^\mu}v(t)$ is a solution of

$$(t\frac{d}{dt} - \alpha)^m u(t) = 0. \tag{54}$$

Proof. We confirm this with the aid of the formula:

$$(t\frac{d}{dt} + \mu - \lambda)^m \frac{d^\mu}{dt^\mu}v(t) = \frac{d}{dt}(t\frac{d}{dt} + \mu - 1 - \lambda)^m \frac{d^{\mu-1}}{dt^{\mu-1}}v(t) = \cdots = \frac{d^\mu}{dt^\mu}(t\frac{d}{dt} - \lambda)^m v(t) = 0. \tag{55}$$

■

By using this lemma, we obtain the following theorem from Theorem 5.3.

Theorem 5.5. Let the condition of Theorem 5.1 be satisfied, and $\epsilon \in \mathbb{R}^0_{>0}$. Then we have k_x series of solutions of Equation (44). The solutions in the kth series are given as follows.
(i) When Re $\alpha_k \geq -1$, if $m_k = 1$, $u_{\alpha_k}(t)t^\epsilon$ is a solution, and if $m_k \geq 2$, we have m_k solutions given by

$$u_{\alpha_k}(t)t^\epsilon, \; u_{\alpha_k}(t)t^\epsilon(\log_e t)^l, \quad l \in \mathbb{Z}_{[1,m_k-1]}. \tag{56}$$

(ii) When Re $\alpha_k < -1$, we put $-\mu_k - 1 = \lfloor \text{Re}\,\alpha_k \rfloor$ and $\alpha_k = -\mu_k - 1 + \lambda_k$, as in Theorem 5.2, and then if $m_k = 1$, we have one solution given by $\frac{d^{\mu_k}}{dt^{\mu_k}}[u_{-1+\lambda_k}(t)t^\epsilon]$, and if $m_k \geq 2$, we have m_k solutions given by

$$\frac{d^{\mu_k}}{dt^{\mu_k}}[u_{-1+\lambda_k}(t)t^\epsilon], \quad \frac{d^{\mu_k}}{dt^{\mu_k}}[u_{-1+\lambda_k}(t)t^\epsilon(\log_e t)^l], \quad l \in \mathbb{Z}_{[1,m_k-1]}. \tag{57}$$

In particular, when $\alpha_k = -1$, if $m_k = 1$, $u_{-1}(t)t^\epsilon$ is a solution, and if $m_k \geq 2$, we have m_k solutions given by

$$u_{-1}(t)t^\epsilon,\ u_{-1}(t)t^\epsilon(\log_e t)^l, \quad l \in \mathbb{Z}_{[1,m_k-1]}, \tag{58}$$

and when $\alpha_k \in \mathbb{Z}_{<-1}$, we put $\mu_k = -\alpha_k - 1$, and then if $m_k = 1$, $\frac{d^{\mu_k}}{dt^{\mu_k}}[u_{-1}(t)t^\epsilon]$ is a solution, and if $m_k \geq 2$, we have m_k solutions given by

$$\frac{d^{\mu_k}}{dt^{\mu_k}}[u_{-1}(t)t^\epsilon], \quad \frac{d^{\mu_k}}{dt^{\mu_k}}[u_{-1}(t)t^\epsilon(\log_e t)^l], \quad l \in \mathbb{Z}_{[1,m_k-1]}. \tag{59}$$

We conclude this section by the following remarks.

Remark 5.2. By comparing Theorem 5.2 with Theorem 5.5, we see that the solutions in Theorems 5.2 are obtained from those in Theorem 5.5, by replacing $\frac{d^{\mu_k}}{dt^{\mu_k}}$ by D^{μ_k}, and adding $\tilde{H}(t)$.

Remark 5.3. In distribution theory, a distribution $\tilde{u}(t)$ is a functional, for which number $\langle \tilde{u}(t), \phi(t)\rangle$ is associated with every testing function $\phi(t)$. Lemma 3.1 shows that when $\phi(t)$ is e^{-st}, the numbers for the distributions in Theorem 5.2 are equal to the AC-Laplace transforms of the corresponding functions in Theorem 5.5.

6 AC-Laplace Transform of $t^{-n+\epsilon}$ and $t^{-n+\epsilon}(\log_e t)^m$ and Their Pseudofunctions for $\epsilon \in \mathbb{C}^0_{\neq 0}$, $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{>0}$

We know that the AC-Laplace transform of function t^{-n} for $n \in \mathbb{Z}_{>0}$ does not exist. In distribution theory, the AC-Laplace transform of pseudofunction Pf $t^{-n}H(t)$ for $n \in \mathbb{Z}_{>0}$, is defined in its place. For instance, in the case of $n = 1$, we consider $\langle t^{-1}\tilde{H}(t-\epsilon), e^{-st}\rangle$ for $\epsilon \in \mathbb{R}$ satisfying $0 < \epsilon \ll 1$, which is evaluated by

$$\begin{aligned}\langle t^{-1}\tilde{H}(t-\epsilon), e^{-st}\rangle &= \int_\epsilon^\infty t^{-1}e^{-st}dt \\ &= \int_0^1 t^{-1}[e^{-st}-1]dt + \log_e \epsilon - \int_0^\epsilon t^{-1}[e^{-st}-1]dt + \int_1^\infty t^{-1}e^{-st}dt. \end{aligned} \tag{60}$$

In this case, the distribution Pf $t^{-1}\tilde{H}(t)$, which corresponds to pseudofunction Pf $t^{-1}H(t)$, is defined by

$$\langle \text{Pf } t^{-1}\tilde{H}(t), e^{-st}\rangle = \int_0^1 t^{-1}[e^{-st}-1]dt + \int_1^\infty t^{-1}e^{-st}dt, \tag{61}$$

which is the sum of the terms of order $O(\epsilon^0)$ in the righthand side of (60); see [7, Section 1.4].

In the following part of this section, $\epsilon \in \mathbb{C}^0_{\neq 0}$, $\epsilon_1 \in \mathbb{C}^0_{\neq 0}$ and $\epsilon_2 \in \mathbb{C}^0_{\neq 0}$.

By replacing z in (28) by ϵ, we obtain

$$\mathcal{L}_H[t^{-1+\epsilon}] = \Gamma(\epsilon)s^{-\epsilon} = \frac{1}{\epsilon} + s\cdot\mathcal{L}[\log_e t] + O(\epsilon). \tag{62}$$

$\mathcal{L}_H[\text{Pf } t^{-1+\epsilon}]$ is then equal to the sum of terms of $O(\epsilon^0)$ on the righthand side of (62), and hence with the aid of (22), we obtain

$$\mathcal{L}_H[\text{Pf } t^{-1+\epsilon}] = s\cdot\mathcal{L}[\log_e t] = -\gamma - \log_e s. \tag{63}$$

From Lemma 4.1, by replacing z_1 by ϵ_1, we have

Lemma 6.1. *Let* $m \in \mathbb{Z}_{>-1}$*. Then*

$$\mathcal{L}_H[t^{-1+\epsilon_1}(\log_e t)^m] = \frac{(-1)^m m!}{\epsilon_1^{m+1}} + \frac{s}{m+1}\mathcal{L}[(\log_e t)^{m+1}] + O(\epsilon_1), \tag{64}$$

$$\mathcal{L}_H[\mathrm{Pf}\ t^{-1+\epsilon_1}(\log_e t)^m] = \frac{s}{m+1}\mathcal{L}[(\log_e t)^{m+1}], \tag{65}$$

where $\mathcal{L}[(\log_e t)^{m+1}]$ *is given in* (24)*. In particular, when* $m = 1$*, we have*

$$\mathcal{L}_H[\mathrm{Pf}\ t^{-1+\epsilon_1}\log_e t] = \frac{1}{2}s \cdot \mathcal{L}[(\log_e t)^2] = \frac{1}{2}(\gamma^2 + \frac{\pi^2}{6} + 2\gamma\log_e s + (\log_e s)^2). \tag{66}$$

Proof. In obtaining the righthand side of (66), we use (23). ■

Formulas corresponding to (63) and (66) are given in [7, Table B.2]. By using (8) for $n = 1$, and third and fourth equalities in (28), we obtain

$$\begin{aligned}\mathcal{L}_H[t^{-2+\epsilon}] &= \Gamma(\epsilon-1)s^{1-\epsilon} = \frac{\Gamma(\epsilon+1)}{\epsilon(\epsilon-1)}s^{1-\epsilon} = \frac{s}{\epsilon-1}\cdot\frac{s}{\epsilon}\mathcal{L}[t^\epsilon] \\ &= -s(1 + \sum_{k=1}^{\infty}\epsilon^k)(\frac{1}{\epsilon} + \sum_{l=1}^{\infty}\frac{s}{l!}\epsilon^{l-1}\mathcal{L}[(\log_e t)^l]) \\ &= -s[\frac{1}{\epsilon} + 1 + \epsilon + s(1+\epsilon)\mathcal{L}[\log_e t] + \frac{1}{2}\epsilon s\cdot\mathcal{L}[(\log_e t)^2]] + O(\epsilon^2). \end{aligned} \tag{67}$$

By (67), with the aid of (22), we obtain

$$\mathcal{L}_H[\mathrm{Pf}\ t^{-2+\epsilon}] = -s - s^2\cdot\mathcal{L}[\log_e t] = s(\gamma + \log_e s - 1). \tag{68}$$

Lemma 6.2. *Let* $m \in \mathbb{Z}_{>-1}$*. Then*

$$\mathcal{L}_H[t^{-2+\epsilon_1}(\log_e t)^m] = -\frac{(-1)^m m!}{\epsilon_1^{m+1}}s - s\cdot m! - s^2\sum_{l=1}^{m+1}\frac{m!}{l!}\mathcal{L}[(\log_e t)^l] + O(\epsilon_1), \tag{69}$$

$$\mathcal{L}_H[\mathrm{Pf}\ t^{-2+\epsilon_1}(\log_e t)^m] = -s\cdot m! - s^2\sum_{l=1}^{m+1}\frac{m!}{l!}\mathcal{L}[(\log_e t)^l], \tag{70}$$

where $\mathcal{L}[(\log_e t)^l]$ *are given in* (22)∼(24)*. In particular, when* $m = 1$*, we have*

$$\begin{aligned}\mathcal{L}_H[\mathrm{Pf}\ t^{-2+\epsilon_1}\log_e t] &= -s(1 + s\cdot\mathcal{L}[\log_e t] + \frac{1}{2}s\cdot\mathcal{L}[(\log_e t)^2]) \\ &= -\frac{1}{2}s(2 - 2\gamma + \gamma^2 + \frac{\pi^2}{6} + 2(-1+\gamma)\log_e s + (\log_e s)^2). \end{aligned} \tag{71}$$

Proof. The lefthand side of (69) is the term of order ϵ_2^m in (27) for $z_1 = \epsilon_1$, $z_2 = \epsilon_2$ and $n = 2$, multiplied by $\frac{m!}{\epsilon_1^m}$. The righthand side is the sum of corresponding terms in (67) for $\epsilon = \epsilon_1(1+\epsilon_2)$. In obtaining (71) for $m = 1$, we may use the last member of (67) in place of the fifth, (22) and (23). Equations (69) and (70) for $m = 0$ are due to (67) and (68). ■

Definition 6.1. In the following theorems, we use $\zeta_1(n)$, $\zeta_2(n)$ and $\psi(n)$ for $n \in \mathbb{Z}_{>0}$. They are $\zeta_1(1) = \zeta_2(1) = 0$, $\psi(1) = -\gamma$, and

$$\zeta_1(n) = \sum_{\nu=1}^{n-1}\frac{1}{\nu}, \quad \psi(n) = -\gamma + \zeta_1(n), \quad \zeta_2(n) = \sum_{\nu=1}^{n-1}\frac{1}{\nu^2}, \quad n \in \mathbb{Z}_{>1}. \tag{72}$$

Theorem 6.1. Let $n \in \mathbb{Z}_{>0}$, and $\epsilon \in \mathbb{C}^0_{\neq 0}$. Then the AC-Laplace transform of $t^{-n+\epsilon}$ is given by

$$\mathcal{L}_H[t^{-n+\epsilon}] = -\frac{(-1)^n s^{n-1}}{(n-1)!}\frac{1}{\epsilon} + \mathcal{L}_H[\text{Pf } t^{-n+\epsilon}] + O(\epsilon), \tag{73}$$

$$\mathcal{L}_H[\text{Pf } t^{-n+\epsilon}] = -\frac{(-1)^n s^{n-1}}{(n-1)!}(\psi(n) - \log_e s). \tag{74}$$

Proof. By using (8) with n replaced by $n-1$, and then the third equality of (28) for $n = 0$, we have

$$\mathcal{L}_H[t^{-n+\epsilon}] = \Gamma(\epsilon - n + 1)s^{n-1-\epsilon} = \frac{\Gamma(\epsilon+1)s^{n-1}s^{-\epsilon}}{\epsilon(\epsilon-1)^-_{n-1}} = \frac{s^{n-1}}{(\epsilon-1)^-_{n-1}} \cdot \frac{s}{\epsilon}\mathcal{L}[t^\epsilon], \tag{75}$$

where $(\epsilon - 1)^-_n$ is defined by (49). We use the following expansion:

$$\frac{1}{(\epsilon-1)^-_{n-1}} = -\frac{(-1)^n}{(n-1)!}(1 + \sum_{k=1}^{\infty} c_k \cdot \epsilon^k), \tag{76}$$

where c_k are constants. Equations in (72) show that the first two constants c_1 and c_2 are expressed as

$$c_1 = \zeta_1(n), \quad c_2 = \frac{1}{2}(\zeta_1(n)^2 + \zeta_2(n)). \tag{77}$$

By using (76) and the third equality of (28) in (75), we obtain

$$\begin{aligned}\mathcal{L}_H[t^{-n+\epsilon}] &= -\frac{(-1)^n s^{n-1}}{(n-1)!}(1 + \sum_{k=1}^{\infty} c_k \cdot \epsilon^k)(\frac{1}{\epsilon} + \sum_{l=1}^{\infty}\frac{s}{l!}\epsilon^{l-1}\mathcal{L}[(\log_e t)^l]) \\ &= -\frac{(-1)^n s^{n-1}}{(n-1)!}(\frac{1}{\epsilon} + c_1 + c_2 \cdot \epsilon + s(1 + c_1 \cdot \epsilon)\mathcal{L}[\log_e t] + \frac{1}{2}\epsilon s \cdot \mathcal{L}[(\log_e t)^2]) + O(\epsilon^2) \\ &= -\frac{(-1)^n s^{n-1}}{(n-1)!}(\frac{1}{\epsilon} + c_1 + s \cdot \mathcal{L}[\log_e t] + \epsilon(c_2 + c_1 \cdot s\mathcal{L}[\log_e t] + \frac{1}{2}s \cdot \mathcal{L}[(\log_e t)^2])) + O(\epsilon^2).\end{aligned} \tag{78}$$

From this equation, we obtain Equations (73) and (74), with the aid of (22), (77) and (72). ■

When $n = 1$, $c_1 = c_2 = 0$, and when $n = 2$, $c_1 = c_2 = 1$. We then confirm that (78) for $n = 1$ and $n = 2$ agree with (62) and (67), respectively.

Formulas corresponding to (63), (68) and (74) are given in [7, Example 8.3-3].

Theorem 6.2. Let $n \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}_{>0}$ and $\epsilon_1 \in \mathbb{C}^0_{\neq 0}$. Then the AC-Laplace transforms of $t^{-n+\epsilon_1}(\log_e t)^m$ and pseudofunction Pf $t^{-n+\epsilon_1}(\log_e t)^m$ are given by

$$\mathcal{L}_H[t^{-n+\epsilon_1}(\log_e t)^m] = -\frac{(-1)^n s^{n-1} m!}{(n-1)!}\frac{(-1)^m}{\epsilon_1^{m+1}} + \mathcal{L}_H[\text{Pf } t^{-n+\epsilon_1}(\log_e t)^m] + O(\epsilon_1), \tag{79}$$

$$\mathcal{L}_H[\text{Pf } t^{-n+\epsilon_1}(\log_e t)^m] = -\frac{(-1)^n s^{n-1} m!}{(n-1)!}(c_{m+1} + s\sum_{l=1}^{m+1}\frac{1}{l!}c_{m+1-l}\mathcal{L}[(\log_e t)^l]), \tag{80}$$

where $\mathcal{L}[(\log_e t)^l]$ are given in (22)∼(24). In particular, when $m = 1$, we have

$$\mathcal{L}_H[t^{-n+\epsilon_1}\log_e t] = \frac{(-1)^n s^{n-1}}{(n-1)!}\frac{1}{\epsilon_1^2} + \mathcal{L}_H[\text{Pf } t^{-n+\epsilon_1}\log_e t] + O(\epsilon_1), \tag{81}$$

$$\mathcal{L}_H[\text{Pf } t^{-n+\epsilon_1}\log_e t] = -\frac{(-1)^n s^{n-1}}{2(n-1)!}(\psi(n)^2 + \zeta_2(n) + \frac{\pi^2}{6} - 2\psi(n)\log_e s + (\log_e s)^2). \tag{82}$$

Proof. The lefthand side of (79) is the term of order ϵ_2^m in (27) for $z_1 = \epsilon_1$ and $z_2 = \epsilon_2$, multiplied by $\frac{m!}{\epsilon_1^m}$. The righthand side is the sum of corresponding terms in (78) for $\epsilon = \epsilon_1(1+\epsilon_2)$. In obtaining (81) with (82) for $m = 1$, we may use the terms of $O(\epsilon_2)$ in the last member of (78). We then obtain (81) with

$$\mathcal{L}_H[\mathrm{Pf}\ t^{-n+\epsilon_1}\log_e t] = -\frac{(-1)^n s^{n-1}}{(n-1)!}(c_2 + c_1 \cdot s \cdot \mathcal{L}[\log_e t] + \frac{1}{2}s \cdot \mathcal{L}[(\log_e t)^2]). \quad (83)$$

By using (77), (22) and (23) on the righthand side of (83), we obtain (82). ■

When $n = 1$, $c_1 = c_2 = 0$, and when $n = 2$, $c_1 = c_2 = 1$. We then confirm that (74) and (82) for $n = 1$ and $n = 2$ agree with (63) and (66), and (68) and (71), respectively.

7 Conclusion

In the present paper, we started the discussion with the function $g_\nu(t)$ defined by (1). The function as a function of ν is an entire function which has zeros at $\nu \in \mathbb{Z}_{<1}$. Because of these zeros, the index law does not hold for the Riemann-Liouville derivative ${}_0D_R$, which is defined by (2). We note that the break of the index law can be remedied in nonstandard analysis, in Section 1.

We give the Laplace transform of $t^{\nu-1}(\log_e t)^m$ for $\nu \in \mathbb{C}_+$ and $m \in \mathbb{Z}_{>0}$ in Section 3, and the AC-Laplace transform of $t^{-1+z}(\log_e t)^m$ for $m \in \mathbb{Z}_{>-1}$ and $z \in \mathbb{C}$ which satisfies $0 < |z| \ll 1$, in Section 4. These formulas are used in Sections 5 and 6.

In Section 5, we recall the therems on the solutions of Euler's differential equation, which are given in [5]. When the solutions are given for the equation in nonstandard analysis, we can give them in the form, from which the solutions in distribution theory are obtained.

In Section 6, the AC-Laplace transforms of functions $t^{-n+\epsilon}$ and $t^{-n+\epsilon}(\log_e t)^m$ for $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{>0}$, and their pseudofunctions are obtained with the aid of nonstandard analysis.

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