Applications of a generalized singular boundary value problem for the exact solutions of some temperature/concentration equations

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Abstract

In the field of fluid mechanics, the temperature distribution and the nanoparticles concentration are usually described by singular boundary value problems (SBVPs). Such SBVPs are also used to describe various models with applications in engineering and other areas. Generally, obtaining the analytic solutions of such kind of problems is a challenge due to the singularity involved in the governing equations. In this paper, a class of SBVPs is analyzed. The solution of this class is analyzed and investigated through developing several theorems and lemmas. In addition, the theoretical results are invested to construct several solutions for various models/problems in fluid mechanics in the literature. Moreover, the published results are recovered as special cases of our analysis.

Keywords: Nanofluid; temperature; Ordinary differential equation; hypergeometric series; exact solution.

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1 Introduction

In this paper, we consider a generalized class of singular boundary value problem (SPVBs) in the form:

$$\tau^2\chi''(\tau) + \left(P\tau + Q\tau^2\right)\chi'(\tau) + (l + R\tau)\,\chi(\tau) = \sigma\,\tau^{a+1}, \tag{1}$$

such that

$$\chi(0) = 0,\ \chi(b) = 1 + \epsilon\chi'(b),\ b \in \mathbb{R} - \{0\},\ a > -1,\ \sigma \in \mathbb{R},\ \epsilon \in \mathbb{R}. \tag{2}$$

Eq. (1) is introduced in this form such that it covered several special cases in the literature. Moreover, the class (1-2) arises in several engineering applications. For example, the coefficients P, Q, l, σ, and R are related to the properties of nanofluids such as density, thermal conductivity, and heat capacitance [1-11]. Besides, a, b, and ϵ are specified according to the final forms of the heat/concentration equations along with the boundary conditions (BCs). Usually, the researchers resort to direct softwares or approximate numerical/analytical methods to solve physical models in finite/infinite domain [1, 12-19]. Although softwares are capable of solving many scientific models in physics and engineering, they can not provide use with a clear picture about the conditions that admit the convergence of the solutions. In addition, the approximate numerical/analytical methods may need a massive computational work to conduct a solution. Moreover, such approximate methods, sometimes, lead to inaccurate results as pointed out in Refs. [20-24]. Regarding, the authors [20] mentioned that there were great differences between their exact results and those approximately obtained in Ref. [25]. Khaled [21] re-investigated the effects of radiation on the MHD Marangoni convection boundary layer over a flat surface via an exact approach. He concluded that the existing results [26] agree with his exact results up to only three/four digits. Also, the authors [22-24] mentioned further remarks on some approximate methods.

In view of the above discussion, the exact solution is always the best and preferred when available for any physical model. Accordingly, the motivation of this paper is to exactly solve the class (1-2). Generally, obtaining analytic solutions of such a class is a challenge due to the singularity involved in the governing equation. In addition, the current paper is of great benefit, not only for researchers in both pure and applied differential equations, but also for researchers in fluid mechanics. The researchers in a such field will be allowed and able to invest the present results to directly obtain the exact solutions for their future models instead of handling each model separately. So, the main goal of this work is to provide the researchers, especially in the field of fluid mechanics, with the direct solution for possible future models describing the temperature/nanoparticles distributions or other phenomena in the form given by the generalized class (1-2).

The paper is organized as follows. In section 2, a theoretical analysis is introduced which includes the proofs of basic theorems and lemmas. Derivation of these theorems and lemmas is a cornerstone to achieve the task of this paper. Section 3 is devoted to discuss the applications of the present results, where several exact solutions for physical problems in the literature are recovered as special cases of our generalized exact solution. In addition, the present results and outcomes are concluded in section 4.

2 Analysis

Theorem 1: The differential equation (1) reduces to $\tau\rho''(\tau)+(P_1+Q\tau)\,\rho'(\tau)+R_1\rho(\tau)=\sigma\tau^{a-\nu}$ under the transformation $\chi(\tau)=\tau^{\nu}\rho(\tau)$, where $P_1=2\nu+P$, $R_1=\nu Q+R$, and $\nu=\frac{1-P\pm\sqrt{(1-P)^2-4l}}{2}$.

Proof: We have from $\chi(\tau)=\tau^{\nu}\rho(\tau)$ that

$$\chi'(\tau)=\tau^{\nu-1}\left(\tau\rho'(\tau)+\nu\rho(\tau)\right), \tag{3}$$

$$\chi''(\tau)=\tau^{\nu-2}\left(\tau^2\rho''(\tau)+2\nu t\rho'(\tau)+\nu\left(\nu-1\right)\rho(\tau)\right). \tag{4}$$

Substituting Eqs. (3-4) into Eq. (1), yields

$$\tau^2\rho''(\tau)+\left((2\nu+P)\,\tau+Q\tau^2\right)\rho'(\tau)+\left(\left(\nu^2-\nu+\nu P+l\right)+(\nu Q+R)\,\tau\right)\rho(\tau)=\sigma\tau^{a-\nu+1}, \tag{5}$$

which implies that

$$\tau\rho''(\tau)+(P_1+Q\tau)\,\rho'(\tau)+R_1\rho(\tau)=\sigma\tau^{a-\nu}, \tag{6}$$

when

$$\nu^2-\nu+\nu P+l=0, \tag{7}$$

and

$$P_1 = 2\nu + P, \qquad R_1 = \nu Q + R. \tag{8}$$

Solving Eq. (7) for ν, we obtain

$$\nu = \frac{1 - P \pm \sqrt{(1-P)^2 - 4l}}{2}, \tag{9}$$

which completes the proof. The rules of choosing the positive/negative sign in Eq. (9) will be discussed later for several applied problems.

Theorem 2: Under the boundary conditions (2) and the constrain $R = -(a+1)Q$, the solution of Eq. (1) is given by

$$\begin{aligned}\chi(\tau) \quad &= \frac{(\tau/b)^{1-\nu-P} \;\; {}_1F_1[-a-\nu-P, 2-2\nu-P, -Q\;\tau] \left(1 - \frac{\sigma b^a (b-\epsilon(a+1))}{(a-\nu+1)(a+\nu+P)}\right)}{\left(1-\epsilon(1-\nu-P)/b\right)\Lambda_1 - \epsilon Q(\nu+a+P)\Lambda_2} + \\ &\frac{\sigma\;\tau^{a+1}}{(a-\nu+1)\,(a+\nu+P)},\end{aligned} \tag{10}$$

such that

$$1-\nu-P > 0, \quad a > -1, \quad (a-\nu+1)\,(a+\nu+P) \neq 0, \tag{11}$$

where Λ_1 and Λ_2 are defined by

$$\Lambda_1 = {}_1F_1[-a-\nu-P, 2-2\nu-P, -Qb], \tag{12}$$

$$\Lambda_2 = {}_1F_1[1-a-\nu-P, 3-2\nu-P, -Qb], \tag{13}$$

and ${}_1F_1$ is Kummer's function.

Proof: Based on theorem 1, the solution $\chi(\tau)$ of Eq. (1) can be directly obtained when the solution $\rho(\tau)$ of Eq. (6) is available. Let $\rho_c(\tau)$ is the complementary solution and $\rho_p(\tau)$ is the particular solution of Eq. (6), accordingly,

$$\rho(\tau) = \rho_c(\tau) + \rho_p(\tau). \tag{14}$$

Following the authors [27], the $\rho_c(\tau)$ is given as

$$\rho_c(t) = \frac{h\ \tau^{\omega_1+\omega_2-1}}{\Gamma(\omega_1+\omega_2)}\ {}_1F_1[\omega_1, \omega_1+\omega_2, -Q\ \tau], \quad \omega_1+\omega_2 > 1, \tag{15}$$

where h is a constant to be determined, and

$$\omega_1 = 1 - P_1 + \frac{R_1}{Q}, \qquad \omega_2 = 1 - \frac{R_1}{Q}. \tag{16}$$

From Eqs. (8) and Eq. (16), we have

$$\omega_1 = 1 - \nu - P + \frac{R}{Q}, \qquad \omega_2 = 1 - \nu - \frac{R}{Q}. \tag{17}$$

Also, the $\rho_p(t)$ of Eq. (6) can be obtained as

$$\rho_p(t) = \frac{\sigma\ \tau^{a-\nu+1}}{(a-\nu+1)\,(a-\nu+P_1)}, \tag{18}$$

such that

$$R_1 = -(a-\nu+1)Q. \tag{19}$$

From Eq. (19) and Eqs. (8), we obtain

$$R = -(a+1)Q. \tag{20}$$

In this case, Eq. (6) reduces to

$$\tau\rho''(\tau) + ((2\gamma+P)+Q\tau)\,\rho'(\tau) - (n-\gamma+1)\,Q\rho(\tau) = \sigma\tau^{a-\nu}. \tag{21}$$

Substituting P_1 in Eqs. (8) into Eq. (18), yields

$$\rho_p(\tau) = \frac{\sigma\ \tau^{a-\nu+1}}{(a-\nu+1)\,(a+\nu+P)}. \tag{22}$$

The general solution of Eq. (21) is obtained from (14) as

$$\rho(\tau) = \frac{h\ \tau^{\omega_1+\omega_2-1}}{\Gamma(\omega_1+\omega_2)}\ {}_1F_1[\omega_1, \omega_1+\omega_2, -Q\ \tau] + \frac{\sigma\ \tau^{a-\nu+1}}{(a-\nu+1)\,(a+\nu+P)}, \tag{23}$$

where ω_1 and ω_2 are finally defined by

$$\omega_1 = -a - \nu - P, \qquad \omega_2 = 2 + a - \nu. \tag{24}$$

Consequently, the solution of the original equation (1), provided that $R = -(a+1)Q$,

$$\tau^2\chi''(\tau) + \left(P\tau + Q\tau^2\right)\chi'(\tau) + (l - (n+1)Q\tau)\,\chi(\tau) = \sigma\tau^{a+1}, \tag{25}$$

is obtained by

$$\chi(\tau) = \frac{h\ \tau^{\nu+\omega_1+\omega_2-1}}{\Gamma(\omega_1+\omega_2)}\ {}_1F_1[\omega_1, \omega_1+\omega_2, -Q\ \tau] + \frac{\sigma\ \tau^{a+1}}{(a-\nu+1)\,(a+\nu+P)}. \tag{26}$$

It is observed from Eq. (26) that $\chi(0) = 0$ is satisfied if

$$\nu + \omega_1 + \omega_2 > 1, \quad a > -1, \quad (a-\nu+1)\,(a+\nu+P) \neq 0. \tag{27}$$

The constant h is determined from the condition $\chi(b) = 1 + \epsilon\chi'(b)$, hence,

$$h = \frac{b^{1-\nu-\omega_1-\omega_2}\Gamma(\omega_1+\omega_2)\left(1 - \frac{\sigma b^a(b-\epsilon(a+1))}{(a-\nu+1)(a+\nu+P)}\right)}{(1-\epsilon(\nu+\omega_1+\omega_2-1)/b)\,\Lambda_1 + (\epsilon Q\omega_1)\Lambda_2}, \tag{28}$$

where

$$\Lambda_1 = {}_1F_1[\omega_1, \omega_1+\omega_2, -Qb], \qquad \Lambda_2 = {}_1F_1[1+\omega_1, 1+\omega_1+\omega_2, -Qb], \tag{29}$$

and the relation:

$$\frac{d}{d\tau}\left({}_1F_1[\omega_1, \omega_1+\omega_2, -Q\tau]\right) = -\left(Q\omega_1\right)\ {}_1F_1[1+\omega_1, 1+\omega_1+\omega_2, -Q\tau], \tag{30}$$

was implemented to calculate h in (28). Inserting (28) into (26), we get

$$\begin{aligned}\chi(\tau) \quad &= \frac{(\tau/b)^{\nu+\omega_1+\omega_2-1}\ {}_1F_1[\omega_1, \omega_1+\omega_2, -Q\ \tau]\left(1 - \frac{\sigma b^a(b-\epsilon(a+1))}{(a-\nu+1)(a+\nu+P)}\right)}{(1-\epsilon(\nu+\omega_1+\omega_2-1)/b)\,\Lambda_1 + (\epsilon Q\omega_1)\Lambda_2} + \\ &\frac{\sigma\ \tau^{a+1}}{(a-\nu+1)\,(a+\nu+P)}.\end{aligned} \tag{31}$$

Inserting ω_1 and ω_2 from Eqs. (24) into Eqs. (31,27,29), we obtain the solution provided by this theorem.

Lemma 1: If $l = 0$ and $R = -(a+1)Q$, the solution of Eq. (1) is given by

$$\begin{aligned}\chi(\tau) \quad &= \frac{(\tau/b)^{1-P}\ {}_1F_1\left[-a-P, 2-P, -Q\ \tau\right]\left(1 - \frac{\sigma b^a(b-\epsilon(a+1))}{(a+1)(a+P)}\right)}{(1-\epsilon(1-P)/b)\,\Lambda_1 - \epsilon Q(a+P)\Lambda_2} + \\ &\frac{\sigma\ \tau^{a+1}}{(a+1)\,(a+P)},\end{aligned} \tag{32}$$

such that

$$1-P>0, \quad a>-1, \quad (a+1)\,(a+P)\neq 0, \tag{33}$$

where Λ_1 and Λ_2 are defined by

$$\Lambda_1 = {}_1F_1\left[-a-P, 2-P, -Qb\right], \qquad \Lambda_2 = {}_1F_1\left[1-a-P, 3-P, -Qb\right]. \tag{34}$$

Proof: At $l=0$ and $R=-(a+1)Q$, Eq. (1) becomes

$$\chi''(\tau)+\left(\frac{P}{\tau}+Q\right)\chi'(\tau)-\left(\frac{(a+1)Q}{\tau}\right)\chi(\tau)=\sigma\tau^{a-1}, \quad a>-1,\ \sigma\in\mathbb{R}, \tag{35}$$

which is equivalent to Eq. (25) when $l=0$. Accordingly, the solution of Eq. (35) is directly obtained from theorem 2, Eqs. (10-13), when ν is calculated at $l=0$. In such case, $\nu=\frac{1-P\pm|1-P|}{2}$ from Eq. (10). For $P<1$ and $P>1$, ν is either $1-P$ or zero. However, $\nu=1-P$ doesn't satisfy $1-\gamma-P>0$ (the first condition in Eq. (11)). Therefore, $\nu=0$ when $l=0$. Thus, the solution given by Eqs. (10-13) reduces to Eqs. (32-34). Moreover, the solution obtained by this lemma agrees with the published one, see Ref. [28] for details.

Lemma 2: If $\sigma=0$, the solution of Eq. (1) is given by

$$\chi(\tau)=\frac{(\tau/b)^{1-\nu-P}\ {}_1F_1\left[1-\nu-P+\frac{R}{Q}, 2-2\nu-P, -Q\,\tau\right]}{(1-\epsilon(1-\nu-P)/b)\,\Lambda_1+\epsilon\,(Q(1-\nu-P)+R)\,\Lambda_2}, \quad 1-\nu-P>0, \tag{36}$$

where

$$\Lambda_1 = {}_1F_1\left[1-\nu-P+\frac{R}{Q}, 2-2\nu-P, -Qb\right], \tag{37}$$

$$\Lambda_2 = {}_1F_1\left[2-\nu-P+\frac{R}{Q}, 3-2\nu-P, -Qb\right]. \tag{38}$$

Proof: At $\sigma=0$, Eq. (1) becomes homogenous and takes the form:

$$\chi''(\tau)+\left(\frac{P}{\tau}+Q\right)\chi'(\tau)+\left(\frac{l}{\tau^2}+\frac{R}{\tau}\right)\chi(\tau)=0, \tag{39}$$

which can be reduced to the following form:

$$\tau\rho''(\tau)+(P_1+Q\tau)\,\rho'(\tau)+R_1\rho(\tau)=0, \tag{40}$$

under the transformation introduced by theorem 1, where P_1 and R_1 are defined by Eqs. (8). The solution of Eq. (40) is

$$\rho(\tau) = \frac{A\ \tau^{\omega_1+\omega_2-1}}{\Gamma(\omega_1+\omega_2)}\ {}_1F_1[\omega_1, \omega_1+\omega_2, -Q\ \tau], \tag{41}$$

and A is a constant to be determined, where ω_1 and ω_2 are given by Eqs. (17). Hence, the solution of Eq. (39) is

$$\chi(\tau) = \frac{A\ \tau^{\nu+\omega_1+\omega_2-1}}{\Gamma(\omega_1+\omega_2)}\ {}_1F_1[\omega_1, \omega_1+\omega_2, -Q\ \tau]. \tag{42}$$

The condition $\chi(0) = 0$ is satisfied when $\nu + \omega_1 + \omega_2 > 1$. Applying $\chi(b) = 1 + \epsilon\chi'(b)$ on Eq. (42), yields

$$A = \frac{b^{1-\nu-\omega_1-\omega_2}\Gamma(\omega_1+\omega_2)}{(1-\epsilon(\nu+\omega_1+\omega_2-1)/b)\,\Lambda_1 + (\epsilon Q\omega_1)\Lambda_2}, \tag{43}$$

where Λ_1 and Λ_2 are given in their general forms by Eqs. (29). Substituting Eq. (43) into Eq. (42) and implementing Eqs. (17) and Eqs. (29), we obtain the solution provided by this lemma.

3 Applications

It is shown in this section that the solution provided by theorem 2 reduces to several solutions in the relevant literature as special cases of the parameters σ, ϵ, a, and b and the coefficients P, Q, l, and R.

3.1 $\epsilon = 0$, $l = 0$, $\sigma \neq 0$

In this case the class (1-2) reduces to the same one of Ref. [29]:

$$\tau\chi''(\tau) + (P + Q\tau)\,\chi'(\tau) - ((n+1)Q)\,\chi(\tau) = \sigma\ \tau^a, \quad \chi(0) = 0, \quad \chi(b) = 1. \tag{44}$$

It was declared in lemma 1 that $l = 0$ leads to $\nu = 0$ and hence the solution provided by lemma 1 can be applied here, in the absence of ϵ. Substituting $\epsilon = 0$ into Eq. (32), we obtain

$$\chi(\tau) = \frac{(\tau/b)^{1-P}\ {}_1F_1[-a-P, 2-P, -Q\ \tau]}{\Lambda_1}\left(1-\frac{\sigma(b)^{a+1}}{(a+1)\,(a+P)}\right)+ \frac{\sigma\ \tau^{a+1}}{(a+1)\,(a+P)}. \tag{45}$$

Inserting Λ_1 defined by Eq. (34) into Eq. (45), yields

$$\chi(\tau) = \frac{(\tau/b)^{1-P}\ {}_1F_1[-a-P, 2-P, -Q\ \tau]}{{}_1F_1[-a-P, 2-P, -Qb]}\left(1-\frac{\sigma(b)^{a+1}}{(a+1)\,(a+P)}\right)+ \frac{\sigma\ \tau^{a+1}}{(a+1)\,(a+P)}, \tag{46}$$

which is the same result of Ref. [29].

3.2 $\epsilon \neq 0$, $l = 0$, $\sigma = 0$, $b = 1$

Here, the system (1-2) reduces to that one studied by [27]

$$\chi''(\tau) + \left(\frac{P}{\tau} + Q\right)\chi'(\tau) + \left(\frac{R}{\tau}\right)\chi(\tau) = 0, \tag{47}$$

such that

$$\chi(0) = 0, \quad \chi(1) = 1 + \epsilon\chi'(1). \tag{48}$$

Here, Eq. (47) is a special case of Eq. (39) when $l = 0$. Therefore, the solution of Eqs. (47-48) is derived from lemma 2 by substituting $\nu - 1 - P$ into Eq. (36). Consequently,

$$\chi(\tau) = \frac{\tau^{1-P}{}_1F_1\left[1-P+\frac{R}{Q}, 2-P, -Q\tau\right]}{(1-\epsilon(1-P)/b)\,\Lambda_1 + \epsilon\,(Q\,(1-P)+R)\,\Lambda_2}, \tag{49}$$

where Λ_1 and Λ_2, given in Eqs. (37-38), become

$$\Lambda_1 = {}_1F_1\left[1-P+\frac{R}{Q}, 2-P, -Qb\right], \qquad \Lambda_2 = {}_1F_1\left[2-P+\frac{R}{Q}, 3-P, -Qb\right]. \tag{50}$$

The results in Eqs. (49-50) are in full agreement with those of Ref. [27].

3.3 $\epsilon = 0$, $l \neq 0$, $\sigma \neq 0$

At the special case $\epsilon = 0$, our solution given by Eq. (10) reduces to

$$\chi(\tau) = \frac{(\tau/b)^{1-\nu-P} \; {}_1F_1[-a-\nu-P, 2-2\nu-P, -Q\,\tau]\left(1-\frac{\sigma b^{a+1}}{(a-\nu+1)(a+\nu+P)}\right)}{\Lambda_1} + \frac{\sigma\,\tau^{a+1}}{(a-\nu+1)\,(a+\nu+P)}. \tag{51}$$

Inserting Λ_1 given by Eq. (12) into Eq. (51), we obtain

$$\chi(\tau) = \frac{(\tau/b)^{1-\nu-P} \; {}_1F_1[-a-\nu-P, 2-2\nu-P, -Q\,\tau]\left(1-\frac{\sigma b^{a+1}}{(a-\nu+1)(a+\nu+P)}\right)}{{}_1F_1[-a-\nu-P, 2-2\nu-P, -Qb]} + \frac{\sigma\,\tau^{a+1}}{(a-\nu+1)\,(a+\nu+P)}, \tag{52}$$

which agrees with the solution in Ref. [30] (see Eq. 3.16 in [30]) as special cases of our equation (52).

4 Conclusion

In this paper, a generalized class of singular BVPs was analyzed and exactly solved. The considered class was of wide applications in nanofluids researches. Some theorems and lemmas were theoretically proven in several cases of the involved coefficients of the generalized governing equation. The present results may be of great interest for researchers, not only in pure/applied differential equations, but also for researchers in fluid mechanics. In view of our results, several existing solutions were derived as special cases. Instead of handling each model separately, researchers in a such field are able to directly construct the exact solutions for their possible future models in fluid mechanics. The current results not only save time and effort for researchers in this field, but also provided the best solutions, which are the exact solutions.

5 Conflict of interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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