On the Sums of Squares of Generalized Tribonacci Numbers: Closed Formulas
of $\sum_{k=0}^{n} x^k W_k^2$

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Abstract. In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_k^2$, $\sum_{k=0}^{n} x^k W_{k+1} W_k$ and $\sum_{k=0}^{n} x^k W_{k+2} W_k$ for the squares of generalized Tribonacci numbers are presented. As special cases, we give summation formulas of Tribonacci, Tribonacci-Lucas, Padovan, Perrin numbers and the other third order recurrence relations. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery. Our work generalize third order recurrence relations.

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Keywords. Sums of squares, third order recurrence, generalized Tribonacci numbers, Padovan numbers, Perrin numbers, Narayana numbers.

1. Introduction

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n \geq 0}$ (or shortly $\{W_n\}_{n \geq 0}$) is defined as follows:

$$(1.1) \quad W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3$$

where $W_0, W_1, W_2$ are arbitrary complex numbers and $r, s, t$ are real numbers. The generalized Tribonacci sequence has been studied by many authors, see for example [2,3,7,10,28,33,42,43,44,64,72,78,79,80,82].

The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{t} W_{-(n-1)} - \frac{s}{t} W_{-(n-2)} + \frac{1}{t} W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer $n$. 


In literature, for example, the following names and notations (see Table 1) are used for the special case of $r, s, t$ and initial values.

Table 1. A few special case of generalized Tribonacci sequences.

<table>
<thead>
<tr>
<th>Sequences (Numbers)</th>
<th>Notation</th>
<th>OEIS [71]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tribonacci</td>
<td>${T_n} = {V_n(0, 1, 1; 1, 1, 1)}$</td>
<td>A000073, A057597</td>
</tr>
<tr>
<td>Tribonacci-Lucas</td>
<td>${K_n} = {V_n(3, 1, 3; 1, 1, 1)}$</td>
<td>A001644, A073145</td>
</tr>
<tr>
<td>third order Pell</td>
<td>${P_n^{(3)}} = {V_n(0, 1, 2; 2, 1, 1)}$</td>
<td>A077939, A077978</td>
</tr>
<tr>
<td>third order Pell-Lucas</td>
<td>${Q_n^{(3)}} = {V_n(3, 2, 6; 2, 1, 1)}$</td>
<td>A276225, A276228</td>
</tr>
<tr>
<td>third order modified Pell</td>
<td>${E_n^{(3)}} = {V_n(0, 1, 1; 2, 1, 1)}$</td>
<td>A077997, A078049</td>
</tr>
<tr>
<td>Padovan (Cordonnier)</td>
<td>${P_n} = {V_n(1, 1, 1; 0, 1, 1)}$</td>
<td>A000931</td>
</tr>
<tr>
<td>Perrin (Padovan-Lucas)</td>
<td>${E_n} = {V_n(3, 0, 2; 0, 1, 1)}$</td>
<td>A001608, A078712</td>
</tr>
<tr>
<td>Padovan-Perrin</td>
<td>${S_n} = {V_n(0, 0, 1; 0, 1, 1)}$</td>
<td>A000931, A176971</td>
</tr>
<tr>
<td>Pell-Padovan</td>
<td>${R_n} = {V_n(1, 1, 1; 0, 2, 1)}$</td>
<td>A066983, A128587</td>
</tr>
<tr>
<td>Jacobsthal-Padovan</td>
<td>${Q_n} = {V_n(1, 1, 1; 0, 1, 2)}$</td>
<td>A159284</td>
</tr>
<tr>
<td>Jacobsthal-Perrin (-Lucas)</td>
<td>${L_n} = {V_n(3, 0, 2; 0, 1, 2)}$</td>
<td>A072328</td>
</tr>
<tr>
<td>Narayana</td>
<td>${N_n} = {V_n(0, 1, 1; 1, 0, 1)}$</td>
<td>A078012</td>
</tr>
<tr>
<td>Narayana-Lucas</td>
<td>${U_n} = {V_n(3, 1, 1; 1, 0, 1)}$</td>
<td>A001609</td>
</tr>
<tr>
<td>Narayana-Perrin</td>
<td>${H_n} = {V_n(3, 0, 2; 1, 0, 1)}$</td>
<td>A077947</td>
</tr>
<tr>
<td>third order Jacobsthal</td>
<td>${J_n^{(3)}} = {V_n(0, 1, 1; 1, 1, 2)}$</td>
<td>A226308</td>
</tr>
<tr>
<td>third order Jacobsthal-Lucas</td>
<td>${j_n^{(3)}} = {V_n(2, 1, 5; 1, 1, 2)}$</td>
<td>A226308</td>
</tr>
<tr>
<td>3-primes</td>
<td>${G_n} = {V_n(0, 1, 2; 2, 3, 5)}$</td>
<td>A073145</td>
</tr>
<tr>
<td>Lucas 3-primes</td>
<td>${H_n} = {V_n(3, 2, 10; 2, 3, 5)}$</td>
<td>A077978</td>
</tr>
<tr>
<td>modified 3-primes</td>
<td>${E_n} = {V_n(0, 1, 1; 2, 3, 5)}$</td>
<td>A276225, A276228</td>
</tr>
</tbody>
</table>

Here OEIS stands for On-line Encyclopedia of Integer Sequences. 3-primes, Lucas 3-primes and modified 3-primes sequences can also be called (named) as Grahaml, Grahaml-Lucas and modified Grahaml sequences, respectively, see [65].

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{k=0}^{n} (-1)^k T_k^2 = \frac{1}{4}((-1)^n (T_{n+3}^2 - 2T_{n+2}^2 + 3T_{n+1}^2 - 2T_{n+1}T_{n+3}) - 1)$$

and

$$\sum_{k=0}^{n} (-1)^k N_k^2 = \frac{1}{3}((-1)^n (N_{n+3}^2 - 2N_{n+2}^2 + 2N_{n+1}^2 - 2N_{n+3}N_{n+1} + 2N_{n+2}N_{n+1}) - 1).$$

In this work, we derive expressions for sums of second powers of generalized Tribonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers.
<table>
<thead>
<tr>
<th>Name of sequence</th>
<th>sums of second powers</th>
<th>sums of third powers</th>
<th>sums of powers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized Fibonacci</td>
<td>[4,5,14,22,23,66]</td>
<td>[12,67,69,70,81]</td>
<td>[6,11,35]</td>
</tr>
<tr>
<td>Generalized Tribonacci</td>
<td>[40,68]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Tetranacci</td>
<td>[36,41]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. An Application of the Sum of the Squares of the Numbers

An application of the sum of the squares of the numbers is circulant matrix. Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized $m$-step Fibonacci sequences require the sum of the squares of the numbers of the sequences. For generalized $m$-step Fibonacci sequences see for example Soykan [53]. If $m = 2$, $m = 3$ and $m = 4$, we get the generalized Fibonacci sequence, generalized Tribonacci sequence and generalized Tetranacci sequence, respectively. Next, we recall some information on circulant (r-circulant, geometric circulant) matrices and Frobenius norm, spectral norm, maximum column length norm and maximum row length norm.

Circulant matrices have been around for a long time and have been extensively used in many scientific areas. In some scientific areas such as image processing, coding theory and signal processing we often encounter circulant matrices. These matrices also have many applications in numerical analysis, optimization, digital image processing, mathematical statistics and modern technology.

Let $n \geq 2$ be an integer and $r$ be any real or complex number. An $n \times n$ matrix $C_r$ is called a $r$-circulant matrix if it of the form

$$C_r = \begin{pmatrix}
    c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
    rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
    rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 
\end{pmatrix}_{n \times n}$$

and the $r$-circulant matrix $C_r$ is denoted by $C_r = \text{Circ}_r(c_0, c_1, \ldots, c_{n-1})$. If $r = 1$ then 1-circulant matrix is called as circulant matrix and denoted by $C = \text{Circ}(c_0, c_1, \ldots, c_{n-1})$.

- Circulant matrix was first proposed by Davis in [9]. This matrix has many interesting properties, and it is one of the most important research subject in the field of the computational and pure mathematics (see for example references given in Table 3). For instance, Shen and Cen [48] studied on the norms of $r$-circulant matrices with Fibonacci and Lucas numbers.
Then, later Kızılıçak and Tuglu [25] defined a new geometric circulant matrix as follows:

\[
C_{r^n} = \begin{pmatrix}
  c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
  r c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
  r^2 c_{n-2} & r c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  r^{n-1} c_1 & r^{n-2} c_2 & r^{n-3} c_3 & \cdots & r c_{n-1} & c_0
\end{pmatrix}_{n \times n}
\]

and then they obtained the bounds for the spectral norms of geometric circulant matrices with the generalized Fibonacci number and Lucas numbers.

When the parameter satisfies \( r = 1 \), we get the classical circulant matrix. See also Polat [34] for the spectral norms of \( r \)-circulant matrices with a type of Catalan triangle numbers.

The Frobenius (or Euclidean) norm and spectral norm of a matrix \( A = (a_{ij})_{m \times n} \in M_{m \times n}(\mathbb{C}) \) are defined respectively as follows:

\[
\|A\|_F = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad \|A\|_2 = \left( \max_{1 \leq i \leq n} |\lambda_i| \right)^{1/2}
\]

where \( \lambda_i \)'s are the eigenvalues of the matrix \( A^*A \) and \( A^* \) is the conjugate of transpose of the matrix \( A \). The maximum column length norm \( c_1(\cdot) \) and the maximum row length norm \( r_1(\cdot) \) of an matrix of order \( n \times n \) are defined as follows:

\[
c_1(A) = \max_{1 \leq j \leq n} \left( \sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad r_1(A) = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}
\]

The following inequality holds for any matrix \( A = M_{n \times n}(\mathbb{C}) \):

\[
\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F.
\]

Calculations of the above norms \( \|A\|_F \), \( \|A\|_2 \), \( c_1(A) \) and \( r_1(A) \) require the sum of the squares of the numbers \( a_{ij} \). As in our case, the numbers \( a_{ij} \) can be chosen as elements of second, third or higher order linear recurrence sequences.

In the following Table 3, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (\( r \)-circulant, geometric circulant, semicirculant) matrices with the generalized \( m \)-step Fibonacci sequences which require sum formulas of second powers of numbers in \( m \)-step Fibonacci sequences \( (m = 2, 3, 4) \).

Table 3. Papers on the norms.
Also linear summing formulas of the generalized $m$-step Fibonacci sequences are required for the computation of various norms of circulant matrices with the generalized $m$-step Fibonacci sequences. We present some works on summing formulas of the numbers in the following Table 4.

Table 4. A few special study of sum formulas.

<table>
<thead>
<tr>
<th>Name of sequence</th>
<th>Papers which deal with summing formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pell and Pell-Lucas</td>
<td>[15],[26,27]</td>
</tr>
<tr>
<td>Generalized Fibonacci</td>
<td>[16,54,55,62,63]</td>
</tr>
<tr>
<td>Generalized Tribonacci</td>
<td>[13,31,56]</td>
</tr>
<tr>
<td>Generalized Tetranacci</td>
<td>[57,58,83]</td>
</tr>
<tr>
<td>Generalized Pentanacci</td>
<td>[59,60]</td>
</tr>
<tr>
<td>Generalized Hexanacci</td>
<td>[61]</td>
</tr>
</tbody>
</table>

3. Main Result

**Theorem 3.1.** Let $x$ be a complex number. If $\Delta = (-t^2x^3 + sx + rt\,x^2 + 1)(r^2x - s^2x^2 + t^2\,x^3 + 2sx + 2rt\,x^2 - 1) \neq 0$ then

(a): $$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta_1}{(-t^2x^3 + sx + rt\,x^2 + 1)(r^2x - s^2x^2 + t^2\,x^3 + 2sx + 2rt\,x^2 - 1)},$$

(b): $$\sum_{k=0}^{n} x^k W_{k+1}W_k = \frac{\Delta_2}{(-t^2x^3 + sx + rt\,x^2 + 1)(r^2x - s^2x^2 + t^2\,x^3 + 2sx + 2rt\,x^2 - 1)},$$

(c): $$\sum_{k=0}^{n} x^k W_{k+2}W_k = \frac{\Delta_3}{(-t^2x^3 + sx + rt\,x^2 + 1)(r^2x - s^2x^2 + t^2\,x^3 + 2sx + 2rt\,x^2 - 1)}.$$
where

\[ \Delta_1 = -x^{n+3}(t^2x^3 + sx + rt^2x - 1)W_{n+3}^2 \\
- x^{n+2}(r^2x + t^2x^3 + sx + r^2t^2x^4 + rt^2x^2 + r^2sx^2 + r^3tx^3 + 2rstx^3 - 1)W_{n+2}^2 \\
- x^{n+1}(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx + r^2t^2x^4 + s^2t^2x^5 + rt^2x^2 + r^2sx^2 + r^3tx^3 + 2rstx^3 - 1)W_1^2 \\
+ x(r^2x + t^2x^3 + sx + r^2t^2x^4 + rt^2x^2 + r^2sx^2 + r^3tx^3 + 2rstx^3 - 1)W_0^2 \\
+ 2x^{n+4}(r + tx)(s + rtx)W_{n+3}W_{n+2} + 2x^{n+4}t(r + stx)W_{n+3}W_{n+1} \\
- 2x^{n+4}t(sx - 1)(s + rtx)W_{n+2}W_{n+1} - 2x^3(r + tx)(s + rtx)W_2W_1 \\
- 2tx^3(r + stx^2)W_2W_0 + 2x^3t(sx - 1)(s + rtx)W_1W_0 \\
\]

and

\[ \Delta_2 = x^{n+3}(r + stx^2)W_{n+3}^2 + x^{n+4}(t + rs)(s + rtx)W_{n+2}^2 + x^{n+4}t^2(r + stx^2)W_{n+1}^2 \\
- x^{n+2}(r^2x + s^2x^2 + t^2x^3 + 2rstx^3 - 1)W_{n+3}W_{n+2} + x^{n+3}t(r^2x - s^2x^2 - t^2x^3 + 1)W_{n+3}W_{n+1} \\
- x^{n+1}(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx + rt^2x^2 + r^2sx^2 + r^3tx^3 \\
- rt^3x^5 - st^2x^4 + 2rstx^3 - rs^2tx^4 - 1)W_{n+2}W_{n+1} - x^2(r + stx^2)W_2^2 \\
- x^3(t + rs)(s + rtx)W_1^2 - x^3t^2(r + stx^2)W_0^2 + x(r^2x + s^2x^2 + t^2x^3 + 2rstx^3 - 1)W_2W_1 \\
- x^2t(r^2x - s^2x^2 - t^2x^3 + 1)W_2W_0 + (r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx \\
+ rt^2x^2 + r^2sx^2 + r^3tx^3 - rt^2x^5 - st^2x^4 + 2rstx^3 - rs^2tx^4 - 1)W_1W_0 \\
\]
and

\[ \Delta_3 = x^{n+3}(s - s^2x + r^2 + rtx)W_{n+3}^2 + x^{n+2}(s - s^2x + r^2t^2x^3 - r^2sx + rt^3x^4 - rs^2tx^3)W_{n+2}^2 \]
\[ + x^{n+4}t^2(s - s^2x + r^2 + rtx)W_{n+1}^2 - x^{n+2}(r + tx)(r^2x - s^2x^2 + t^2x^3 - 1)W_{n+3}W_{n+2} \]
\[ - x^{n+1}(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx + r^2sx^2 - st^2x^4 + 2rstdx^3 - 1)W_{n+3}W_{n+1} \]
\[ + x^{n+2}(t(sx - 1)(r^2x - s^2x^2 + t^2x^3 - 1)W_{n+1} - x^2(s - s^2x + r^2 + rtx)W_{n+2}^2 \]
\[ + x(-(s - s^2x - r^2t^2x^3 + r^2sx - rt^3x^4 + rs^2tx^3)W_1^2 - x^3t^2(s - s^2x + r^2 + rtx)W_0^2 \]
\[ + x(r + tx)(r^2x - s^2x^2 + t^2x^3 - 1)W_2W_1 \]
\[ + (r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx + r^2sx^2 - st^2x^4 + 2rstdx^3 - 1)W_2W_0 \]
\[ - xt(sx - 1)(r^2x - s^2x^2 + t^2x^3 - 1)W_1W_0. \]

**Proof.** First, we obtain \( \sum_{k=0}^{n} W_k^2 \). Using the recurrence relation

\[ W_{n+3} = rW_{n+2} + sW_{n+1} + tW_n \]

or

\[ tW_n = W_{n+3} - rW_{n+2} - sW_{n+1} \]

i.e.

\[ t^2W_n^2 = (W_{n+3} - rW_{n+2} - sW_{n+1})^2 = W_{n+3}^2 + r^2W_{n+2}^2 + s^2W_{n+1}^2 - 2rW_{n+3}W_{n+2} - 2sW_{n+3}W_{n+1} + 2rsW_{n+2}W_{n+1} \]

we obtain

\[ t^2x^nW_n^2 = x^nW_{n+3}^2 + r^2x^nW_{n+2}^2 + s^2x^nW_{n+1}^2 - 2rxx^nW_{n+3}W_{n+2} - 2sxx^nW_{n+3}W_{n+1} + 2rsxx^nW_{n+2}W_{n+1} \]
\[ t^2x^{n-1}W_{n-1}^2 = x^{n-1}W_{n+2}^2 + r^2x^{n-1}W_{n+1}^2 + s^2x^{n-1}W_n^2 - 2rx^{n-1}W_{n+2}W_{n+1} \]
\[ - 2sx^{n-1}W_{n+2}W_n + 2rsx^{n-1}W_{n+1}W_n \]
\[ t^2x^{n-2}W_{n-2}^2 = x^{n-2}W_{n+1}^2 + r^2x^{n-2}W_n^2 + s^2x^{n-2}W_{n-1}^2 - 2rx^{n-2}W_{n+1}W_n \]
\[ - 2sx^{n-2}W_{n+1}W_{n-1} + 2rsx^{n-2}W_nW_{n-1} \]
\[ : \]
\[ t^2x^2W_2^2 = x^2W_2^2 + r^2x^2W_4^2 + s^2x^2W_3^2 - 2rx^2W_5W_4 - 2sx^2W_5W_3 + 2rsx^2W_4W_3 \]
\[ t^2xW_1^2 = xW_1^2 + r^2xW_3^2 + s^2xW_2^2 - 2rxW_4W_3 - 2sxW_4W_2 + 2rsW_4W_2 \]
\[ t^2x^0W_0^2 = x^0W_2^2 + r^2x^0W_4^2 + s^2x^0W_3^2 - 2rx^0W_5W_4 - 2sx^0W_5W_3 + 2rsx^0W_4W_3 \]
If we add the equations by side by, we get

\[(3.1)\quad t^2 \sum_{k=0}^{n} x^k W_k^2 = \sum_{k=3}^{n+3} x^{k-3} W_k^2 + r^2 \sum_{k=2}^{n+2} x^{k-2} W_k^2 + s^2 \sum_{k=1}^{n+1} x^{k-1} W_k^2 - 2r \sum_{k=2}^{n+2} x^{k-2} W_{k+1}^2 W_k - 2s \sum_{k=1}^{n+1} x^{k-1} W_{k+2}^2 W_k + 2rs \sum_{k=1}^{n+1} x^{k-1} W_{k+1} W_k\]

Next we obtain \(\sum_{k=0}^{n} W_{k+1}^2 W_k\). Multiplying the both side of the recurrence relation

\[t W_n = W_{n+3} - r W_{n+2} - s W_{n+1}\]

by \(W_{n+1}\) we get

\[t W_{n+1} W_n = W_{n+3} W_{n+1} - r W_{n+2} W_{n+1} - s W_{n+2}^2 W_{n+1} \cdot\]

Then using last recurrence relation, we obtain

\[tx^n W_{n+1} W_n = x^n W_{n+3} W_{n+1} - r x^n W_{n+2} W_{n+1} - s x^n W_{n+1}^2\]

\[tx^{n-1} W_{n} W_{n-1} = x^{n-1} W_{n+2} W_n - r x^{n-1} W_{n+1} W_n - s x^{n-1} W_n^2\]

\[tx^{n-2} W_{n-1} W_{n-2} = x^{n-2} W_{n+1} W_{n-1} - r x^{n-2} W_n W_{n-1} - s x^{n-2} W_{n-1}^2\]

\[\vdots\]

\[tx^2 W_2 W_2 = x^2 W_3 W_3 - r x^2 W_4 W_3 - s x^2 W_3^2\]

\[tx W_1 W_1 = x W_2 W_2 - r x W_3 W_2 - s x W_2^2\]

\[tx^0 W_1 W_0 = x^0 W_2 W_1 - r x^0 W_2 W_1 - s x^0 W_1^2\]

If we add the equations by side by, we get

\[(3.2)\quad t \sum_{k=0}^{n} x^k W_k = \sum_{k=1}^{n+1} x^{k-1} W_{k+2} W_k - r \sum_{k=1}^{n+1} x^{k-1} W_{k+1} W_k - s \sum_{k=1}^{n+1} x^{k-1} W_k^2\]

Next we obtain \(\sum_{k=2}^{n} W_{k+2} W_k\). Multiplying the both side of the recurrence relation

\[t W_n = W_{n+3} - r W_{n+2} - s W_{n+1}\]

by \(W_{n+2}\) we get

\[t W_{n+2} W_n = W_{n+3} W_{n+2} - r W_{n+2}^2 - s W_{n+2} W_{n+1}\]

Then using last recurrence relation, we obtain
If we add the equations by side by, we get

\[\begin{align*}
tx^n W_{n+2} W_n & = x^n W_{n+3} W_{n+2} - rx^n W_{n+2}^2 - sx^n W_{n+2} W_{n+1} \\
tx^{n-1} W_{n+1} W_{n-1} & = x^{n-1} W_{n+2} W_{n+1} - rx^{n-1} W_{n+1}^2 - sx^{n-1} W_{n+1} W_{n} \\
tx^{n-2} W_{n} W_{n-2} & = x^{n-2} W_{n+1} W_{n} - rx^{n-2} W_{n}^2 - sx^{n-2} W_{n} W_{n-1} \\
\vdots
\end{align*}\]

If we add the equations by side by, we get

\[\begin{align*}
tx^n W_{n+2} W_n & = x^n W_{n+3} W_{n+2} - rx^n W_{n+2}^2 - sx^n W_{n+2} W_{n+1} \\
tx^{n-1} W_{n+1} W_{n-1} & = x^{n-1} W_{n+2} W_{n+1} - rx^{n-1} W_{n+1}^2 - sx^{n-1} W_{n+1} W_{n} \\
tx^{n-2} W_{n} W_{n-2} & = x^{n-2} W_{n+1} W_{n} - rx^{n-2} W_{n}^2 - sx^{n-2} W_{n} W_{n-1} \\
\vdots
\end{align*}\]

If we add the equations by side by, we get

\[\begin{align*}
tx^n W_{n+2} W_n & = x^n W_{n+3} W_{n+2} - rx^n W_{n+2}^2 - sx^n W_{n+2} W_{n+1} \\
tx^{n-1} W_{n+1} W_{n-1} & = x^{n-1} W_{n+2} W_{n+1} - rx^{n-1} W_{n+1}^2 - sx^{n-1} W_{n+1} W_{n} \\
tx^{n-2} W_{n} W_{n-2} & = x^{n-2} W_{n+1} W_{n} - rx^{n-2} W_{n}^2 - sx^{n-2} W_{n} W_{n-1} \\
\vdots
\end{align*}\]

Solving the system (3.1)-(3.2)-(3.3), the results in (a), (b) and (c) follow.

### 4. Specific Cases

In this section, for the specific cases of \(x\), we present the closed form solutions (identities) of the sums \(\sum_{k=0}^{n} x^k W_k^2\), \(\sum_{k=0}^{n} x^k W_{k+1} W_k\) and \(\sum_{k=0}^{n} x^k W_{k+2} W_k\) for the special case of sequence \(\{W_n\}\).

#### 4.1. The case \(x = 1\)

In this subsection we consider the special case \(x = 1\). See also [68] for some third order recurrence relations (with the sum starting from 0).

Taking \(r = s = t = 1\) in Theorem 3.1, we obtain the following Proposition.

**Proposition 4.1.** If \(r = s = t = 1\) then for \(n \geq 0\) we have the following formulas:

(a): \(\sum_{k=0}^{n} W_k^2 = \frac{1}{4}(-W_{n+3}^2 - 4W_{n+2}^2 - 5W_{n+1}^2 + 4W_{n+3} W_{n+2} + 2W_{n+3}^2 W_{n+1} + W_2 + 4W_2^2 + 5W_0^2 - 4W_2 W_1 - 2W_2 W_0)\).

(b): \(\sum_{k=0}^{n} W_{k+1} W_k = \frac{1}{4}(W_{n+3}^2 + 2W_{n+2}^2 + W_{n+1}^2 - 2W_{n+3} W_{n+2} - 2W_{n+2} W_{n+1} - W_2^2 - 2W_2^2 - W_0^2 + 2W_2 W_1 + 2W_2 W_0)\).

(c): \(\sum_{k=0}^{n} W_{k+2} W_k = \frac{1}{4}(W_{n+3}^2 + W_{n+1}^2 - 2W_{n+3} W_{n+1} - W_2^2 - W_0^2 + 2W_2 W_0)\).

From the above proposition, we have the following Corollary which gives sum formulas of Tribonacci numbers (take \(W_n = T_n\) with \(T_0 = 0, T_1 = 1, T_2 = 1\)).

**Corollary 4.2.** For \(n \geq 0\), Tribonacci numbers have the following properties:

(a): \(\sum_{k=0}^{n} T_k^2 = \frac{1}{4}(-T_{n+3}^2 - 4T_{n+2}^2 - 5T_{n+1}^2 + 4T_{n+3} T_{n+2} + 2T_{n+3}^2 T_{n+1} + 1)\).

(b): \(\sum_{k=0}^{n} T_{k+1} T_k = \frac{1}{4}(T_{n+3}^2 + 2T_{n+2}^2 + T_{n+1}^2 - 2T_{n+3} T_{n+2} - 2T_{n+2} T_{n+1} - 1)\).

(c): \(\sum_{k=0}^{n} T_{k+2} T_k = \frac{1}{4}(T_{n+3}^2 + T_{n+1}^2 - 2T_{n+3} T_{n+1} - 1)\).

Taking \(W_n = K_n\) with \(K_0 = 3, K_1 = 1, K_2 = 3\) in the above Proposition, we have the following Corollary which presents sum formulas of Tribonacci-Lucas numbers.
Corollary 4.3. For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:

(a): $\sum_{k=0}^{n} K_k^2 = \frac{1}{4}(-K_{n+3}^2 - 4K_{n+2}^2 - 5K_{n+1}^2 + 4K_{n+3}K_{n+2} + 2K_{n+3}K_{n+1} + 28)$.

(b): $\sum_{k=0}^{n} K_{k+1}K_k = \frac{1}{4}(K_{n+3}^2 + 2K_{n+2}^2 - 2K_{n+3}K_{n+2} - 2K_{n+2}K_{n+1} - 8)$.

(c): $\sum_{k=0}^{n} K_{k+2} K_k = \frac{1}{4}(K_{n+3}^2 + K_{n+1}^2 - 2K_{n+3}K_{n+1})$.

Taking $r = 2, s = 1, t = 1$ in Theorem 3.1, we obtain the following Proposition.

Proposition 4.4. If $r = 2, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:

(a): $\sum_{k=0}^{n} W_k^2 = \frac{1}{9}(-W_{n+3}^2 - 10W_{n+2}^2 + 2W_{n+3}W_{n+1} + 6W_{n+3}W_{n+2} + W_2^2 + 9W_1^2 + 10W_0^2 - 6W_2W_1 - 2W_2W_0)$.

(b): $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{9}(W_{n+3}^2 + 3W_{n+2}^2 - W_{n+1}^2 - 3W_{n+3}W_{n+2} + W_{n+3}W_{n+1} - 6W_{n+2}W_{n+1} - W_2^2 - 3W_1^2 - W_0^2 + 3W_2W_1 - W_2W_0 + 6W_1W_0)$.

(c): $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{9}(2W_{n+3}^2 - 3W_{n+3}W_{n+2} - 4W_{n+3}W_{n+1} - 2W_2^2 - 2W_0^2 + 3W_2W_1 + 4W_2W_0)$.

From the last Proposition, we have the following Corollary which gives sum formulas of Third-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1$).

Corollary 4.5. For $n \geq 0$, third-order Pell numbers have the following properties:

(a): $\sum_{k=0}^{n} P_k^2 = \frac{1}{9}(-P_{n+3}^2 - 10P_{n+2}^2 + 9P_{n+3}P_{n+1} + 6P_{n+3}P_{n+2} + 1)$.

(b): $\sum_{k=0}^{n} P_{k+1}P_k = \frac{1}{9}(P_{n+3}^2 + 3P_{n+2}^2 - P_{n+1}^2 - 3P_{n+3}P_{n+2} + P_{n+3}P_{n+1} - 6P_{n+2}P_{n+1} - 1)$.

(c): $\sum_{k=0}^{n} P_{k+2}P_k = \frac{1}{9}(2P_{n+3}^2 + 2P_{n+1}^2 - 3P_{n+3}P_{n+2} - 4P_{n+3}P_{n+1} - 2)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$ in the last Proposition, we have the following Corollary which presents sum formulas of third-order Pell-Lucas numbers.

Corollary 4.6. For $n \geq 0$, third-order Pell-Lucas numbers have the following properties:

(a): $\sum_{k=0}^{n} Q_k^2 = \frac{1}{9}(-Q_{n+3}^2 - 10Q_{n+2}^2 + 9Q_{n+3}Q_{n+1} + 6Q_{n+3}Q_{n+2} + 54)$.

(b): $\sum_{k=0}^{n} Q_{k+1}Q_k = \frac{1}{9}(Q_{n+3}^2 + 3Q_{n+2}^2 - Q_{n+1}^2 - 3Q_{n+3}Q_{n+2} + Q_{n+3}Q_{n+1} - 6Q_{n+2}Q_{n+1} - 3)$.

(c): $\sum_{k=0}^{n} Q_{k+2}Q_k = \frac{1}{9}(2Q_{n+3}^2 + 2Q_{n+1}^2 - 3Q_{n+3}Q_{n+2} - 4Q_{n+3}Q_{n+1} + 18)$.

From the last Proposition, we have the following Corollary which gives sum formulas of third-order modified Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 4.7. For $n \geq 0$, third-order modified Pell numbers have the following properties:

(a): $\sum_{k=0}^{n} E_k^2 = \frac{1}{9}(-E_{n+3}^2 - 10E_{n+2}^2 + 9E_{n+3}E_{n+1} + 6E_{n+3}E_{n+2} + 4)$.

(b): $\sum_{k=0}^{n} E_{k+1}E_k = \frac{1}{9}(E_{n+3}^2 + 3E_{n+2}^2 - E_{n+1}^2 - 3E_{n+3}E_{n+2} + E_{n+3}E_{n+1} - 6E_{n+2}E_{n+1} - 1)$.

(c): $\sum_{k=0}^{n} E_{k+2}E_k = \frac{1}{9}(2E_{n+3}^2 + 2E_{n+1}^2 - 3E_{n+3}E_{n+2} - 4E_{n+3}E_{n+1} + 1)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 3.1, we obtain the following Proposition.
PROPOSITION 4.8. If \( r = 0, s = 1, t = 1 \) then for \( n \geq 0 \) we have the following formulas:

\[
\text{(a): } \sum_{k=0}^{n} W_k^2 = -W_{n+3}^2 - W_{n+2}^2 - 2W_{n+1}^2 + 2W_{n+3}W_{n+1} + 2W_n^2 + W_2^2 + W_1^2 - 2W_2W_1.
\]

\[
\text{(b): } \sum_{k=0}^{n} W_{k+1}W_k = W_{n+3}^2 + W_{n+2}^2 + W_{n+1}^2 - W_{n+3}W_{n+2} - W_{n+3}W_{n+1} - W_2^2 - W_1^2 - W_0^2 + W_2W_1.
\]

\[
\text{(c): } \sum_{k=0}^{n} W_{k+2}W_k = W_{n+3}W_{n+2} - W_2W_1.
\]

From the last Proposition, we have the following Corollary which gives sum formulas of Padovan numbers (take \( W_n = P_n \) with \( P_0 = 1, P_1 = 1, P_2 = 1 \)).

COROLLARY 4.9. For \( n \geq 0 \), Padovan numbers have the following properties:

\[
\text{(a): } \sum_{k=0}^{n} P_k^2 = -P_{n+3}^2 - P_{n+2}^2 - 2P_{n+1}^2 + 2P_{n+3}P_{n+1} + 2P_{n+3}P_{n+2}.
\]

\[
\text{(b): } \sum_{k=0}^{n} P_{k+1}P_k = P_{n+3}^2 + P_{n+2}^2 + P_{n+1}^2 - P_{n+3}P_{n+2} - P_{n+3}P_{n+1} - 1.
\]

\[
\text{(c): } \sum_{k=0}^{n} P_{k+2}P_k = P_{n+3}P_{n+2} - 1.
\]

Taking \( W_n = E_n \) with \( E_0 = 3, E_1 = 0, E_2 = 2 \) in the last Proposition, we have the following Corollary which presents sum formulas of Perrin numbers.

COROLLARY 4.10. For \( n \geq 0 \), Perrin numbers have the following properties:

\[
\text{(a): } \sum_{k=0}^{n} E_k^2 = -E_{n+3}^2 - E_{n+2}^2 - 2E_{n+1}^2 + 2E_{n+3}E_{n+1} + 2E_{n+3}E_{n+2} + 10.
\]

\[
\text{(b): } \sum_{k=0}^{n} E_{k+1}E_k = E_{n+3}^2 + E_{n+2}^2 + E_{n+1}^2 - E_{n+3}E_{n+2} - E_{n+3}E_{n+1} - 7.
\]

\[
\text{(c): } \sum_{k=0}^{n} E_{k+2}E_k = E_{n+3}E_{n+2}.
\]

From the last Proposition, we have the following Corollary which gives sum formulas of Padovan-Perrin numbers (take \( W_n = S_n \) with \( S_0 = 0, S_1 = 0, S_2 = 1 \)).

COROLLARY 4.11. For \( n \geq 0 \), Padovan-Perrin numbers have the following properties:

\[
\text{(a): } \sum_{k=0}^{n} S_k^2 = -S_{n+3}^2 - S_{n+2}^2 - 2S_{n+1}^2 + 2S_{n+3}S_{n+1} + 2S_{n+3}S_{n+2} + 1.
\]

\[
\text{(b): } \sum_{k=0}^{n} S_{k+1}S_k = S_{n+3}^2 + S_{n+2}^2 + S_{n+1}^2 - S_{n+3}S_{n+2} - S_{n+3}S_{n+1} - 1.
\]

\[
\text{(c): } \sum_{k=0}^{n} S_{k+2}S_k = S_{n+3}S_{n+2}.
\]

Taking \( r = 0, s = 2, t = 1 \) in Theorem 3.1, we obtain the following theorem.

THEOREM 4.12. If \( r = 0, s = 2, t = 1 \) then for \( n \geq 0 \) we have the following formulas:

\[
\text{(a): } \sum_{k=0}^{n} W_k^2 = \frac{1}{2}((2n + 11) W_{n+3}^2 + (2n + 9) W_{n+2}^2 + (2n + 11) W_{n+1}^2 - 4(n + 5) W_{n+3}W_{n+2} - 4(n + 6) W_{n+3}W_{n+1} + 4(n + 6) W_{n+2}W_{n+1} - 9W_2^2 - 7W_1^2 - 9W_0^2 + 16W_2W_1 + 20W_2W_0 - 20W_1W_0).
\]

\[
\text{(b): } \sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{2}(-2(n + 5) W_{n+3}^2 - 2(n + 4) W_{n+2}^2 - 2(n + 6) W_{n+1}^2 + (4n + 19) W_{n+3}W_{n+2} + (4n + 23) W_{n+3}W_{n+1} - (4n + 23) W_{n+2}W_{n+1} + 8W_2^2 + 6W_1^2 + 10W_0^2 + 15W_2W_1 - 19W_2W_0 + 19W_1W_0).
\]

\[
\text{(c): } \sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{2}(2(n + 5) W_{n+3}^2 + 2(n + 4) W_{n+2}^2 + 2(n + 6) W_{n+1}^2 - (4n + 17) W_{n+3}W_{n+2} - (4n + 23) W_{n+3}W_{n+1} + (4n + 21) W_{n+2}W_{n+1} - 8W_2^2 - 6W_1^2 - 10W_0^2 + 13W_2W_1 + 19W_2W_0 - 17W_1W_0).
\]
Proof.

(a): We use Theorem 3.1 (a). If we set $r = 0, s = 2, t = 1$ in Theorem 3.1 (a) then we have

$$
\sum_{k=0}^{n} W_k^2 = \frac{g_1(x)}{-(x-1)(x+1)(-x+x^2-1)(-3x+x^2+1)}
$$

where

$$
g_1(x) = (-x^3+2x-1)x^{n+3}W_{n+3}^2 - (x^3+2x-1)x^{n+2}W_{n+2}^2 - (4x^5-7x^3+4x^2+2x-1)x^{n+1}W_{n+1}^2 + 4x^{n+5}W_{n+3}W_{n+2} + 4x^{n+6}W_{n+3}^2W_{n+1} + 4(2x-1)x^{n+4}W_{n+2}W_{n+1} + x^2(x^3+2x-1)W_2^2 + x(x^3+2x-1)W_1^2 + (4x^5-7x^3+4x^2+2x-1)W_0^2 - 4x^4W_2W_1 - 4x^5W_2W_0 + 4(2x-1)x^3W_1W_0)
$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$
\sum_{k=0}^{n} W_k^2 = \frac{d}{dx}(g_1(x)) \Big|_{x=1} = \frac{1}{2}(2n+11)W_{n+3}^2 + (2n+9)W_{n+2}^2 + (2n+11)W_{n+1}^2 - 4(n+5)W_{n+3}W_{n+2} - 4(n+6)W_{n+3}W_{n+1}W_{n+1} - 9W_2^2 - 7W_1^2 - 4W_0^2 + 16W_1W_0 + 20W_2W_0 - 20W_1W_0).
$$

(b): We use Theorem 3.1 (b). If we set $r = 0, s = 2, t = 1$ in Theorem 3.1 (b) then we have

$$
\sum_{k=0}^{n} W_{k+1}W_k = \frac{g_2(x)}{-(x-1)(x+1)(-x+x^2-1)(-3x+x^2+1)}
$$

where

$$
g_2(x) = (2x^2x^{n+3}W_{n+3}^2 + 2x^{n+4}W_{n+2}^2 + 2x^2x^{n+4}W_{n+1}^2 - (x^3+4x^2-1)x^{n+2}W_{n+3}W_{n+2} - (x^3+4x^2-1)x^{n+3}W_{n+3}W_{n+1} + (2x^4+7x^3-4x^2-2x+1)x^{n+1}W_{n+2}W_{n+1} - 2x^4W_2^2 - 2x^3W_1^2 - 2x^5W_0^2 + x(x^3+4x^2-1)W_2W_1 + (x^3+4x^2-1)x^2W_2W_0 - (2x^4+7x^3-4x^2-2x+1)W_1W_0)
$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$
\sum_{k=0}^{n} W_{k+1}W_k = \frac{d}{dx}(g_2(x)) \Big|_{x=1} = \frac{1}{2}(-2(n+5)W_{n+3}^2 - 2(n+4)W_{n+2}^2 - 2(n+6)W_{n+1}^2 + (4n+19)W_{n+3}W_{n+2} + (4n+23)W_{n+3}W_{n+1} - (4n+23)W_{n+2}W_{n+1} + 8W_2^2 + 6W_1^2 + 10W_0^2 - 15W_1W_0 - 19W_2W_0 - 19W_1W_0).
$$
(c): We use Theorem 3.1 (c). If we set \( r = 0, s = 2, t = 1 \) in Theorem 3.1 (c) then we have

\[
\sum_{k=0}^{n} W_{k+2} W_k = \frac{g_3(x)}{- (x-1)(x+1)(-x + x^2 - 1)(-3x + x^2 + 1)}
\]

where

\[
g_3(x) = (-4x - 2)x^{n+3}W_{n+3} - x^{n+2}(4x - 2)W_{n+2}^2 - x^{n+4}(4x - 2)W_{n+1}^2 - (x^3 + 4x^2 + 1)x^{n+3}W_{n+3}W_{n+2} + (2x^4 + 7x^3 - 4x^2 - 2x + 1)x^{n+1}W_{n+3}W_{n+1} - (2x - 1) (x^3 + 4x^2 + 1)x^{n+2}W_{n+2}W_{n+1} + (4x - 2)x^2W_2W_1 + (4x - 2)x^3W_0^2 - (x^3 + 4x^2 + 1)x^2W_2W_1 - (2x^4 + 7x^3 - 4x^2 - 2x + 1)W_2W_0 + x(2x - 1)(x^3 + 4x^2 + 1)W_1W_0
\]

For \( x = 1 \), the right hand side of the above sum formula is an indeterminate form. Now, we can use L’Hospital rule. Then we get

\[
\sum_{k=0}^{n} W_{k+2} W_k = \left. \frac{d}{dx} \left( \frac{g_3(x)}{- (x-1)(x+1)(-x + x^2 - 1)(-3x + x^2 + 1)} \right) \right|_{x=1}
\]

\[
= \frac{1}{2} (2(n + 5)W_{n+3}^2 + 2(n + 4)W_{n+2}^2 + 2(n + 6)W_{n+1}^2 - (4n + 17)W_{n+3}W_{n+2}
- (4n + 23)W_{n+3}W_{n+1} + (4n + 21)W_{n+2}W_{n+1} - 8W_2^2 - 6W_1^2 - 10W_0^2 + 13W_2W_1 + 19W_2W_0 - 17W_1W_0).
\]

From the last theorem, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take \( W_n = R_n \) with \( Q_0 = 1, R_1 = 1, R_2 = 1 \)).

**Corollary 4.13.** For \( n \geq 0 \), Pell-Padovan numbers have the following properties:

(a): \( \sum_{k=0}^{n} R_k^2 = \frac{1}{2}((2n + 11)R_{n+3}^2 + (2n + 9)R_{n+2}^2 + (2n + 11)R_{n+1}^2 - 4(n + 5)R_{n+1}R_{n+2} - 4(n + 6)R_{n+3}R_{n+1} - 4(n + 6)R_{n+2}R_{n+1}) \).

(b): \( \sum_{k=0}^{n} R_{k+1}R_k = \frac{1}{2}(-2(n + 5)R_{n+3}^2 - 2(n + 4)R_{n+2}^2 - 2(n + 6)R_{n+1}^2 + (4n + 19)R_{n+1}R_{n+2} + (4n + 23)R_{n+3}R_{n+1} - (4n + 23)R_{n+2}R_{n+1} + 9) \).

(c): \( \sum_{k=0}^{n} R_{k+2}R_k = \frac{1}{2}(2(n + 5)R_{n+3}^2 + 2(n + 4)R_{n+2}^2 + 2(n + 6)R_{n+1}^2 - (4n + 17)R_{n+3}R_{n+2} - (4n + 23)R_{n+3}R_{n+1} + (4n + 21)R_{n+2}R_{n+1} - 9) \).

Taking \( W_n = C_n \) with \( C_0 = 3, C_1 = 0, C_2 = 2 \) in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

**Corollary 4.14.** For \( n \geq 0 \), Pell-Perrin numbers have the following properties:

(a): \( \sum_{k=0}^{n} C_k^2 = \frac{1}{2}((2n + 11)C_{n+3}^2 + (2n + 9)C_{n+2}^2 + (2n + 11)C_{n+1}^2 - 4(n + 5)C_{n+1}C_{n+2} - 4(n + 6)C_{n+3}C_{n+1} + 4(n + 6)C_{n+2}C_{n+1} + 3) \).

(b): \( \sum_{k=0}^{n} C_{k+1}C_k = \frac{1}{2}(-2(n + 5)C_{n+3}^2 - 2(n + 4)C_{n+2}^2 - 2(n + 6)C_{n+1}^2 + (4n + 19)C_{n+3}C_{n+2} + (4n + 23)C_{n+3}C_{n+1} - (4n + 23)C_{n+2}C_{n+1} + 8) \).
(c): \( \sum_{k=0}^{n} C_{k+2}C_k = \frac{1}{2}(2(n+5)C_{n+3}^2 + 2(n+4)C_{n+2}^2 + 2(n+6)C_{n+1}^2 - (4n+17)C_{n+3}C_{n+2} - (4n+23)C_{n+3}C_{n+1} + (4n+21)C_{n+2}C_{n+1} - 8) \).

Taking \( r = 0, s = 1, t = 2 \) in Theorem 3.1, we obtain the following proposition.

**Proposition 4.15.** If \( r = 0, s = 1, t = 2 \) then for \( n \geq 0 \) we have the following formulas:

(a): \( \sum_{k=0}^{n} W_k^2 = \frac{1}{2}(W_{n+3}^2 + W_{n+2}^2 + 2W_{n+1}^2 - W_{n+3}W_{n+2} - 2W_{n+3}W_{n+1} - W_2^2 - W_2^2 - 2W_0^2 + W_2W_1 + 2W_2W_0) \).

(b): \( \sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{4}(-W_{n+3}^2 + W_{n+2}^2 - 4W_{n+1}^2 + 2W_{n+3}W_{n+2} + 4W_{n+3}W_{n+1} + W_2^2 + W_1^2 + 4W_0^2 - 2W_2W_1 - 4W_2W_0) \).

(c): \( \sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{2}(W_{n+3}W_{n+2} - W_1W_2) \).

From the last Proposition, we have the following Corollary which gives sum formulas of Jacobsthal-Padovan numbers (take \( W_n = Q_n \) with \( Q_0 = 1, Q_1 = 1, Q_2 = 1 \)).

**Corollary 4.16.** For \( n \geq 0 \), Jacobsthal-Padovan numbers have the following properties:

(a): \( \sum_{k=0}^{n} Q_k^2 = \frac{1}{2}(Q_{n+3}^2 + Q_{n+2}^2 + 2Q_{n+1}^2 - Q_{n+3}Q_{n+2} - 2Q_{n+3}Q_{n+1} - 1) \).

(b): \( \sum_{k=0}^{n} Q_{k+1}Q_k = \frac{1}{4}(-Q_{n+3}^2 - Q_{n+2}^2 - 4Q_{n+1}^2 + 2Q_{n+3}Q_{n+2} + 4Q_{n+3}Q_{n+1}) \).

(c): \( \sum_{k=0}^{n} Q_{k+2}Q_k = \frac{1}{2}(Q_{n+3}Q_{n+2} - 1) \).

Taking \( W_n = L_n \) with \( L_0 = 3, L_1 = 0, L_2 = 2 \) in the last Proposition, we have the following Corollary which presents sum formulas of Jacobsthal-Perrin numbers.

**Corollary 4.17.** For \( n \geq 0 \), Jacobsthal-Perrin numbers have the following properties:

(a): \( \sum_{k=0}^{n} L_k^2 = \frac{1}{2}(L_{n+3}^2 + L_{n+2}^2 + 2L_{n+1}^2 - L_{n+3}L_{n+2} - 2L_{n+3}L_{n+1} - 10) \).

(b): \( \sum_{k=0}^{n} L_{k+1}L_k = \frac{1}{4}(-L_{n+3}^2 - L_{n+2}^2 - 4L_{n+1}^2 + 2L_{n+3}L_{n+2} + 4L_{n+3}L_{n+1} + 16) \).

(c): \( \sum_{k=0}^{n} L_{k+2}L_k = \frac{1}{2}L_{n+3}L_{n+2} \).

Taking \( r = 1, s = 0, t = 1 \) in Theorem 3.1, we obtain the following Proposition.

**Proposition 4.18.** If \( r = 1, s = 0, t = 1 \) then for \( n \geq 0 \) we have the following formulas:

(a): \( \sum_{k=0}^{n} W_k^2 = \frac{1}{3}(-W_{n+3}^2 - 4W_{n+2}^2 - 4W_{n+1}^2 + 4W_{n+3}W_{n+2} + 2W_{n+3}W_{n+1} + 2W_{n+2}W_{n+1} + W_2^2 + 4W_2^2 - 4W_0^2 + 2W_2W_1 - 2W_2W_0) \).

(b): \( \sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{3}(W_{n+3}^2 + W_{n+2}^2 + W_{n+1}^2 - W_{n+3}W_{n+2} + W_{n+3}W_{n+1} - 2W_{n+2}W_{n+1} - W_2^2 - W_0^2 + W_2W_1 - W_2W_0 + 2W_2W_0) \).

(c): \( \sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{3}(2W_{n+3}^2 + 2W_{n+2}^2 + 2W_{n+1}^2 - 2W_{n+3}W_{n+2} - W_{n+3}W_{n+1} - W_{n+2}W_{n+1} - 2W_2^2 - 2W_0^2 + 2W_2W_1 + W_2W_0 + W_1W_0) \).

From the last proposition, we have the following corollary which gives sum formulas of Narayana numbers (take \( W_n = N_n \) with \( N_0 = 0, N_1 = 1, N_2 = 1 \)).
Corollary 4.19. For $n \geq 0$, Narayana numbers have the following properties:

(a): $\sum_{k=0}^{n} N_{k}^{2} = \frac{1}{3}(-N_{n+3}^{2} - 4N_{n+2}^{2} - 4N_{n+1}^{2} + 4N_{n+3}N_{n+2} + 2N_{n+3}N_{n+1} + 2N_{n+2}N_{n+1} + 1)$.

(b): $\sum_{k=0}^{n} N_{k+1}N_{k} = \frac{1}{3}(N_{n+3}^{2} + N_{n+2}^{2} + N_{n+1}^{2} - N_{n+3}N_{n+2} + N_{n+3}N_{n+1} - 2N_{n+2}N_{n+1} - 1)$.

(c): $\sum_{k=0}^{n} N_{k+2}N_{k} = \frac{1}{3}(2N_{n+3}^{2} + 2N_{n+2}^{2} + 2N_{n+1}^{2} - 2N_{n+3}N_{n+2} - N_{n+3}N_{n+1} - N_{n+2}N_{n+1} - 2)$.

Taking $W_{n} = U_{n}$ with $U_{0} = 3, U_{1} = 1, U_{2} = 1$ in the last proposition, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

Corollary 4.20. For $n \geq 0$, Narayana-Lucas numbers have the following properties:

(a): $\sum_{k=0}^{n} U_{k}^{2} = \frac{1}{3}(-U_{n+3}^{2} - 4U_{n+2}^{2} - 4U_{n+1}^{2} + 4U_{n+3}U_{n+2} + 2U_{n+3}U_{n+1} + 2U_{n+2}U_{n+1} + 25)$.

(b): $\sum_{k=0}^{n} U_{k+1}U_{k} = \frac{1}{3}(U_{n+3}^{2} + U_{n+2}^{2} + U_{n+1}^{2} - U_{n+3}U_{n+2} + U_{n+3}U_{n+1} - 2U_{n+2}U_{n+1} - 7)$.

(c): $\sum_{k=0}^{n} U_{k+2}U_{k} = \frac{1}{3}(2U_{n+3}^{2} + 2U_{n+2}^{2} + 2U_{n+1}^{2} - 2U_{n+3}U_{n+2} - U_{n+3}U_{n+1} - U_{n+2}U_{n+1} - 14)$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $W_{n} = H_{n}$ with $H_{0} = 3, H_{1} = 0, H_{2} = 2$).

Corollary 4.21. For $n \geq 0$, Narayana-Perrin numbers have the following properties:

(a): $\sum_{k=0}^{n} H_{k}^{2} = \frac{1}{3}(-H_{n+3}^{2} - 4H_{n+2}^{2} - 4H_{n+1}^{2} + 4H_{n+3}H_{n+2} + 2H_{n+3}H_{n+1} + 2H_{n+2}H_{n+1} + 28)$.

(b): $\sum_{k=0}^{n} H_{k+1}H_{k} = \frac{1}{3}(H_{n+3}^{2} + H_{n+2}^{2} + H_{n+1}^{2} - H_{n+3}H_{n+2} + H_{n+3}H_{n+1} - 2H_{n+2}H_{n+1} - 19)$.

(c): $\sum_{k=0}^{n} H_{k+2}H_{k} = \frac{1}{3}(2H_{n+3}^{2} + 2H_{n+2}^{2} + 2H_{n+1}^{2} - 2H_{n+3}H_{n+2} - H_{n+3}H_{n+1} - H_{n+2}H_{n+1} - 20)$.

Taking $r = 1, s = 1, t = 2$ in Theorem 3.1, we obtain the following theorem.

Theorem 4.22. If $r = 1, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:

(a): $\sum_{k=0}^{n} W_{k}^{2} = \frac{1}{65}((6n + 35)W_{n+3}^{2} + (18n + 90)W_{n+2}^{2} + (24n + 101)W_{n+1}^{2} + 6(3n + 16)W_{n+3}W_{n+2} - 4(3n + 16)W_{n+3}W_{n+1} + 12W_{n+2}W_{n+1} - 29W_{n+2}^{2} - 72W_{n+1}^{2} - 77W_{n+1}^{2} + 78W_{n+1} + 52W_{n+2} - 12W_{n+1} + 0)$.

(b): $\sum_{k=0}^{n} W_{k+1}W_{k} = \frac{1}{65}(-(3n + 13)W_{n+3}^{2} - 3(3n + 14)W_{n+2}^{2} - 4(3n + 16)W_{n+1}^{2} + 9n + 45)W_{n+3}W_{n+2} + 2(3n + 22)W_{n+3}W_{n+1} - 27W_{n+2}W_{n+1} + 10W_{n+2}^{2} + 33W_{n+1}^{2} + 52W_{n+2}^{2} - 36W_{n+1} - 38W_{n+2} + 27W_{n+1} + 0)$.

(c): $\sum_{k=0}^{n} W_{k+2}W_{k} = \frac{1}{65}(-(3n + 10)W_{n+3}^{2} - (9n + 54)W_{n+2}^{2} - 4(3n + 13)W_{n+1}^{2} + (9n + 57)W_{n+3}W_{n+2} + (6n + 17)W_{n+3}W_{n+1} - 6W_{n+2}W_{n+1} + 7W_{n+2}^{2} + 45W_{n+1}^{2} + 40W_{n+1}^{2} - 48W_{n+1} - 11W_{n+2} + 6W_{n+1} + 0)$.

Proof.

(a): We use Theorem 3.1 (a). If we set $r = 1, s = 1, t = 2$ in Theorem 3.1 (a) then we have

$$\sum_{k=0}^{n} x^{k}W_{k}^{2} = \frac{g_{4}(x)}{-(x - 1)(4x - 1)(x + x^{2} + 1)(2x + 4x^{2} + 1)}.$$
where
\[
g_4(x) = -x^{n+3}(4x^3 + 2x^2 + x - 1)W^2_{n+3} - (4x^4 + 10x^3 + 3x^2 + 2x - 1)x^{n+2}W^2_{n+2} \\
- (4x^5 + 2x^4 + 13x^3 + 4x^2 + 2x - 1)x^{n+1}W^2_{n+1} + 2(2x + 1)^2x^{n+4}W^2_{n+3}W_{n+2} \\
+ 4(2x^2 + 1)x^{n+4}W_{n+3}W_{n+1} - 4(x - 1)(2x + 1)x^{n+4}W_{n+2}W_{n+1} \\
+ x^2(4x^3 + 2x^2 + x - 1)W^2_2 + x(4x^4 + 10x^3 + 3x^2 + 2x - 1)W^2_1 \\
+ (4x^5 + 2x^4 + 13x^3 + 4x^2 + 2x - 1)W^2_0 - 2(2x + 1)^2x^3W_2W_1 \\
- 4(2x^2 + 1)x^3W_2W_0 + 4(2x + 1)(x - 1)x^3W_1W_0.
\]

For \( x = 1 \), the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get
\[
\sum_{k=0}^n W^2_k = \left. \frac{d}{dx} \left( g_4(x) \right) \right|_{x=1} \\
= \frac{1}{63}((6n + 35)W^2_{n+3} + (18n + 90)W^2_{n+2} + (24n + 101)W^2_{n+1} - 6(3n + 16)W_{n+3}W_{n+2} \\
- 4(3n + 16)W_{n+3}W_{n+1} + 12W_{n+2}W_{n+1} - 29W_2^2 - 72W_2 - 77W_0^2 + 78W_2W_1 \\
+ 52W_2W_0 - 12W_1W_0).
\]

(b): We use Theorem 3.1 (b). If we set \( r = 1, s = 1, t = 2 \) in Theorem 3.1 (b) then we have
\[
\sum_{k=0}^n x^k W^2_{k+1} W_k = \left. \frac{g_5(x)}{-(x - 1)(4x - 1)(x + x^2 + 1)(2x + 4x^2 + 1)} \right|_{x=1}
\]
where
\[
g_5(x) = (2x^2 + 1)x^{n+3}W^2_{n+3} + 3(2x + 1)x^{n+4}W^2_{n+2} + 4(2x^2 + 1)x^{n+4}W^2_{n+1} \\
- (8x^3 + 2x + x - 1)x^{n+2}W_{n+3}W_{n+2} + 2(-4x^3 - x^2 + x + 1)x^{n+3}W_{n+3}W_{n+1} \\
- (-8x^5 - 6x^4 + 9x^3 + 4x^2 + 2x - 1)x^{n+1}W_{n+2}W_{n+1} - (2x^2 + 1)x^2W_2^2 \\
- 3(2x + 1)x^3W_1^2 - 4(2x^2 + 1)x^3W_0^2 + x(8x^3 + x^2 + x + 1)W_2W_1 \\
- 2(-4x^3 - x^2 + x + 1)x^2W_2W_0 + (-8x^5 - 6x^4 + 9x^3 + 4x^2 + 2x - 1)W_1W_0.
\]

For \( x = 1 \), the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get
\[
\sum_{k=0}^n W_{k+1} W_k = \left. \frac{d}{dx} \left( g_5(x) \right) \right|_{x=1} \\
= \frac{1}{63}(-(3n + 13)W^2_{n+3} - 3(3n + 14)W^2_{n+2} - 4(3n + 16)W^2_{n+1} \\
+ (9n + 45)W_{n+3}W_{n+2} + 2(3n + 22)W_{n+3}W_{n+1} - 27W_{n+2}W_{n+1} \\
+ 10W_2^2 + 33W_1^2 + 52W_0^2 - 36W_2W_1 - 38W_2W_0 + 27W_1W_0).
\]
We use Theorem 3.1 (c). If we set \( r = 1, s = 1, t = 2 \) in Theorem 3.1 (c) then we have
\[
\sum_{k=0}^{n} x^k W_{k+2} W_k = \frac{g_0(x)}{-(x-1)(4x-1)(x^2+1)(2x+4x^2+1)}
\]
where
\[
g_0(x) = (x+2)x^{n+3}W_{n+3}^2 + (8x^4 + 2x^3 - 2x + 1)x^{n+2}W_{n+3}^2 + 4(x+2)x^{n+4}W_{n+3}^2
\]
\[-(2x+1)(4x^3 - x^2 + x - 1)x^{n+2}W_{n+3}W_{n+2} - (4x^4 + 7x^3 + 2x^2 + 2x - 1)x^{n+1}W_{n+3}W_{n+1}
\]+ 2(x-1)(4x^3 - x^2 + x - 1)x^{n+2}W_{n+2}W_{n+1} - (x+2)x^2W_2^2 - x(8x^4 + 2x^3 - 2x + 1)W_1^2
\]- 4(x+2)x^3W_0^2 + x(2x+1)(4x^3 - x^2 + x - 1)W_2W_1
\]+ (-4x^4 + 7x^3 + 2x^2 + 2x - 1)W_2W_0 - 2x(x-1)(4x^3 - x^2 + x - 1)W_1W_0.

For \( x = 1 \), the right hand side of the above sum formula is an indeterminate form. Now, we can use L’Hospital rule. Then we get
\[
\sum_{k=0}^{n} W_{k+2} W_k = \frac{d}{dx}(g_0(x)) \bigg|_{x=1}
\]
\[= \frac{1}{63}(-3n+10)W_{n+3}^2 - (9n+54)W_{n+2}^2 - 4(3n+13)W_{n+1}^2
\]+ (9n+57)W_{n+3}W_{n+2} + (6n+17)W_{n+3}W_{n+1} - 6W_{n+2}W_{n+1}
\]+ 7W_2^2 + 45W_1^2 + 40W_0^2 - 48W_2W_1 - 11W_2W_0 + 6W_1W_0.

From the last theorem, we have the following corollary which gives sum formulas of third order Jacobsthal numbers (take \( W_n = J_n \) with \( J_0 = 0, J_1 = 1, J_2 = 1 \)).

**Corollary 4.23.** For \( n \geq 0 \), third order Jacobsthal numbers have the following properties:

(a): \( \sum_{k=0}^{n} J_k^2 = \frac{1}{63}((6n+35)J_{n+3}^2 + (18n+90)J_{n+2}^2 + (24n+101)J_{n+1}^2 - 6(3n+16)J_{n+3}J_{n+2} - 4(3n+16)J_{n+3}J_{n+1} + 12J_{n+2}J_{n+1} - 23) \)

(b): \( \sum_{k=0}^{n} J_{k+1}J_k = \frac{1}{63}((-3n+13)J_{n+3}^2 - 3(3n+14)J_{n+2}^2 - 4(3n+16)J_{n+1}^2 + (9n+45)J_{n+3}J_{n+2} + 2(3n+22)J_{n+3}J_{n+1} - 27J_{n+2}J_{n+1} + 7) \)

(c): \( \sum_{k=0}^{n} J_{k+2}J_k = \frac{1}{63}((-3n+10)J_{n+3}^2 - (9n+54)J_{n+2}^2 - 4(3n+13)J_{n+1}^2 + (9n+57)J_{n+3}J_{n+2} + (6n+17)J_{n+3}J_{n+1} - 6J_{n+2}J_{n+1} + 4) \)

Taking \( W_n = J_n \) with \( j_0 = 2, j_1 = 1, j_2 = 5 \) in the last theorem, we have the following corollary which presents sum formulas of third order Jacobsthal-Lucas numbers.

**Corollary 4.24.** For \( n \geq 0 \), third order Jacobsthal-Lucas numbers have the following properties:

(a): \( \sum_{k=0}^{n} J_k^2 = \frac{1}{63}((6n+35)J_{n+3}^2 + (18n+90)J_{n+2}^2 + (24n+101)J_{n+1}^2 - 6(3n+16)J_{n+3}J_{n+2} - 4(3n+16)J_{n+3}J_{n+1} + 12J_{n+2}J_{n+1} - 219) \)

(b): \( \sum_{k=0}^{n} J_{k+1}J_k = \frac{1}{63}((-3n+13)J_{n+3}^2 - 3(3n+14)J_{n+2}^2 - 4(3n+16)J_{n+1}^2 + (9n+45)J_{n+3}J_{n+2} + 2(3n+22)J_{n+3}J_{n+1} - 27J_{n+2}J_{n+1} - 15) \)
(c): \( \sum_{k=0}^{n} j_{k+2}j_k = \frac{1}{45}(-3n + 10)j_{n+3}^2 - (9n + 54)j_{n+2}^2 - 4(3n + 13)j_{n+1}^2 + (9n + 57)j_{n+3}j_{n+2} + (6n + 17)j_{n+3}j_{n+1} - 6j_{n+2}j_{n+1} + 42). \\

Taking \( r = 2, s = 3, t = 5 \) in Theorem 3.1, we obtain the following proposition.

**Proposition 4.25.** If \( r = 2, s = 3, t = 5 \) then for \( n \geq 0 \) we have the following formulas:

(a): \( \sum_{k=0}^{n} W_k^2 = \frac{1}{455}(37W_{n+3}^2 + 253W_{n+2}^2 + 430W_{n+1}^2 - 182W_{n+3}W_{n+2} - 170W_{n+3}W_{n+1} + 260W_{n+2}W_{n+1} - 37W_3^2 - 253W_2^2 - 430W_1^2 + 182W_2W_1 + 170W_2W_0 - 260W_1W_0). \\
(b): \sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{495}(-17W_{n+3}^2 - 143W_{n+2}^2 - 425W_{n+1}^2 + 97W_{n+3}W_{n+2} + 145W_{n+3}W_{n+1} - 280W_{n+2}W_{n+1} + 17W_3^2 + 143W_2^2 + 425W_1^2 - 97W_2W_1 - 145W_2W_0 - 280W_1W_0). \\
(c): \sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{495}(-8W_{n+3}^2 - 242W_{n+2}^2 - 200W_{n+1}^2 + 133W_{n+3}W_{n+2} + 10W_{n+3}W_{n+1} - 190W_{n+2}W_{n+1} + 8W_3^2 + 242W_2^2 + 200W_1^2 - 133W_2W_1 - 10W_2W_0 + 190W_1W_0). \\

From the last proposition, we have the following corollary which gives sum formulas of 3-primes numbers (take \( W_n = G_n \) with \( G_0 = 0, G_1 = 1, G_2 = 2 \)).

**Corollary 4.26.** For \( n \geq 0 \), 3-primes numbers have the following properties:

(a): \( \sum_{k=0}^{n} G_k^2 = \frac{1}{495}(37G_{n+3}^2 + 253G_{n+2}^2 + 430G_{n+1}^2 - 182G_{n+3}G_{n+2} - 170G_{n+3}G_{n+1} + 260G_{n+2}G_{n+1} - 37). \\
(b): \sum_{k=0}^{n} G_{k+1}G_k = \frac{1}{495}(-17G_{n+3}^2 - 143G_{n+2}^2 - 425G_{n+1}^2 + 97G_{n+3}G_{n+2} + 145G_{n+3}G_{n+1} - 280G_{n+2}G_{n+1} + 17). \\
(c): \sum_{k=0}^{n} G_{k+2}G_k = \frac{1}{495}(-8G_{n+3}^2 - 242G_{n+2}^2 - 200G_{n+1}^2 + 133G_{n+3}G_{n+2} + 10G_{n+3}G_{n+1} - 190G_{n+2}G_{n+1} + 8). \\

Taking \( G_n = H_n \) with \( H_0 = 3, H_1 = 2, H_2 = 10 \) in the last proposition, we have the following corollary which presents sum formulas of Lucas 3-primes numbers.

**Corollary 4.27.** For \( n \geq 0 \), Lucas 3-primes numbers have the following properties:

(a): \( \sum_{k=0}^{n} H_k^2 = \frac{1}{495}(37H_{n+3}^2 + 253H_{n+2}^2 + 430H_{n+1}^2 - 182H_{n+3}H_{n+2} - 170H_{n+3}H_{n+1} + 260H_{n+2}H_{n+1} - 1402). \\
(b): \sum_{k=0}^{n} H_{k+1}H_k = \frac{1}{495}(-17H_{n+3}^2 - 143H_{n+2}^2 - 425H_{n+1}^2 + 97H_{n+3}H_{n+2} + 145H_{n+3}H_{n+1} - 280H_{n+2}H_{n+1} + 1487). \\
(c): \sum_{k=0}^{n} H_{k+2}H_k = \frac{1}{495}(-8H_{n+3}^2 - 242H_{n+2}^2 - 200H_{n+1}^2 + 133H_{n+3}H_{n+2} + 10H_{n+3}H_{n+1} - 190H_{n+2}H_{n+1} + 1748). \\

From the last proposition, we have the following corollary which gives sum formulas of modified 3-primes numbers (take \( H_n = E_n \) with \( E_0 = 0, E_1 = 1, E_2 = 1 \)).

**Corollary 4.28.** For \( n \geq 0 \), modified 3-primes numbers have the following properties:
(a): \( \sum_{k=0}^{n} E_{k}^2 = \frac{1}{495}(37E_{n+3}^2+253E_{n+2}^2+430E_{n+1}^2-182E_{n+3}E_{n+2}-170E_{n+3}E_{n+1}+260E_{n+2}E_{n+1} - 108). \)

(b): \( \sum_{k=0}^{n} E_{k+1}E_{k} = \frac{1}{495}(-17E_{n+3}^2-143E_{n+2}^2-425E_{n+1}^2+97E_{n+3}E_{n+2}+145E_{n+3}E_{n+1}-280E_{n+2}E_{n+1} + 63). \)

(c): \( \sum_{k=0}^{n} E_{k+2}E_{k} = \frac{1}{495}(-8E_{n+3}^2-242E_{n+2}^2-200E_{n+1}^2+133E_{n+3}E_{n+2}+10E_{n+3}E_{n+1}-190E_{n+2}E_{n+1} + 117). \)

4.2. The case \( x = -1 \). In this subsection we consider the special case \( x = -1 \).

In this section, we present the closed form solutions (identities) of the sums \( \sum_{k=0}^{n} (-1)^k W_k^2 \), \( \sum_{k=0}^{n} (-1)^k W_{k+2}W_k \) and \( \sum_{k=0}^{n} (-1)^k W_{k+1}W_k \) for the specific case of the sequence \( \{W_n\} \).

Taking \( r = s = t = 1 \) in Theorem 3.1, we obtain the following proposition.

**Proposition 4.29.** If \( r = s = t = 1 \) then for \( n \geq 0 \) we have the following formulas:

(a): \( \sum_{k=0}^{n} (-1)^k W_k^2 = \frac{1}{4}(((-1)^n (W_{n+3}^2 - 2W_{n+2}^2 + 3W_{n+1}^2 - 2W_{n+1}W_{n+2} + W_{n+2}^2 - 2W^2 + 3W_0^2 - 2W_0W_2)). \)

(b): \( \sum_{k=0}^{n} (-1)^k W_{k+1}W_k = \frac{1}{4}(((-1)^n (W_{n+3}^2 - W_{n+2}^2 - 2W_{n+1}W_{n+2} + 2W_{n+2}W_{n+1} + W_2^2 - W_0^2 - 2W_1W_2 + 2W_1W_0)). \)

(c): \( \sum_{k=0}^{n} (-1)^k W_{k+2}W_k = \frac{1}{4}(((-1)^n (W_{n+3}^2 - 2W_{n+2}^2 - W_{n+1}^2 + 2W_{n+3}W_{n+1} - 4W_{n+2}W_{n+1}) + W_2^2 - 2W_1^2 - W_0^2 + 2W_2W_0 - 4W_1W_0)). \)

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take \( W_n = T_n \) with \( T_0 = 0, T_1 = 1, T_2 = 1 \)).

**Corollary 4.30.** For \( n \geq 0 \), Tribonacci numbers have the following properties:

(a): \( \sum_{k=0}^{n} (-1)^k T_k^2 = \frac{1}{4}(((-1)^n (T_{n+3}^2 - 2T_{n+2}^2 + 3T_{n+1}^2 - 2T_{n+1}T_{n+2} ) - 1). \)

(b): \( \sum_{k=0}^{n} (-1)^k T_{k+1}T_k = \frac{1}{4}(((-1)^n (T_{n+3}^2 - T_{n+2}^2 - 2T_{n+3}T_{n+2} + 2T_{n+2}T_{n+1} ) - 1). \)

(c): \( \sum_{k=0}^{n} (-1)^k T_{k+2}T_k = \frac{1}{4}(((-1)^n (T_{n+3}^2 - 2T_{n+2}^2 - T_{n+1}^2 + 2T_{n+3}T_{n+1} - 4T_{n+2}T_{n+1} ) - 1). \)

Taking \( W_n = K_n \) with \( K_0 = 3, K_1 = 1, K_2 = 3 \) in the above proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

**Corollary 4.31.** For \( n \geq 0 \), Tribonacci-Lucas numbers have the following properties:

(a): \( \sum_{k=0}^{n} (-1)^k K_k^2 = \frac{1}{4}(((-1)^n (K_{n+3}^2 - 2K_{n+2}^2 + 3K_{n+1}^2 - 2K_{n+1}K_{n+2} ) + 16). \)

(b): \( \sum_{k=0}^{n} (-1)^k K_{k+1}K_k = \frac{1}{4}(((-1)^n (K_{n+3}^2 - K_{n+2}^2 - 2K_{n+3}K_{n+2} + 2K_{n+2}K_{n+1}). \)

(c): \( \sum_{k=0}^{n} (-1)^k K_{k+2}K_k = \frac{1}{4}(((-1)^n (K_{n+3}^2 - 2K_{n+2}^2 - K_{n+1}^2 + 2K_{n+3}K_{n+1} - 4K_{n+2}K_{n+1} ) + 4). \)

Taking \( r = 2, s = 1, t = 1 \) in Theorem 3.1, we obtain the following proposition.

**Proposition 4.32.** If \( r = 2, s = 1, t = 1 \) then for \( n \geq 0 \) we have the following formulas:

(a): \( \sum_{k=0}^{n} (-1)^k W_k^2 = \frac{1}{16}(((-1)^n (W_{n+3}^2 - 9W_{n+2}^2 + 14W_{n+1}^2 + 2W_{n+3}W_{n+2} + 4W_{n+2}W_{n+1} - 6W_{n+3}W_{n+1}) + W_2^2 - 9W_1^2 + 14W_0^2 + 2W_2W_1 - 6W_2W_0 + 4W_1W_0). \)
(b): \(\sum_{k=0}^{n}(-1)^kW_{k+1}W_k = \frac{1}{15}((-1)^n(W_{n+3}^2+9W_{n+2}^2-2W_{n+1}^2-6W_{n+2}^2-4W_{n+3}W_{n+2}-2W_{n+2}W_{n+1}+4W_{n+2}W_{n+1})+W_{n+2}^2-6W_{n+1}^2-30W_{n+1}^2-W_{n+2}W_{n+1}+4W_{n+2}W_{n+1})\).

(c): \(\sum_{k=0}^{n}(-1)^kW_{k+2}W_k = \frac{1}{15}((-1)^n(4W_{n+3}^2-6W_{n+2}^2-4W_{n+2}^2-7W_{n+3}W_{n+2}+6W_{n+3}W_{n+1}-14W_{n+2}W_{n+1})+4W_{n+2}^2-6W_{n+1}^2-4W_{n+1}^2-7W_{n+2}W_{n+1}+6W_{n+2}W_{n+1}-14W_{n+1}W_{n+1}).\)

From the last proposition, we have the following corollary which gives sum formulas of third-order Pell numbers (take \(W_n = P_n\) with \(P_0 = 0, P_1 = 1, P_2 = 1\)).

**Corollary 4.33.** For \(n \geq 0\), third-order Pell numbers have the following properties:

(a): \(\sum_{k=0}^{n}(-1)^kP_{k+1}^2 = \frac{1}{15}((-1)^n(P_{n+3}^2+9P_{n+2}^2+12P_{n+1}+2P_{n+2}P_{n+1}+4P_{n+2}P_{n+1}-6P_{n+3}P_{n+1})-1).\)

(b): \(\sum_{k=0}^{n}(-1)^kP_{k+1}P_k = \frac{1}{15}((-1)^n(P_{n+3}^2+9P_{n+2}^2+12P_{n+1}+2P_{n+2}P_{n+1}+4P_{n+2}P_{n+1}-6P_{n+3}P_{n+1})-1).\)

(c): \(\sum_{k=0}^{n}(-1)^kP_{k+2}P_k = \frac{1}{15}((-1)^n(4P_{n+3}^2-6P_{n+2}^2-4P_{n+2}^2-7P_{n+3}P_{n+2}+6P_{n+3}P_{n+1}-14P_{n+2}P_{n+1})-1).\)

Taking \(W_n = Q_n\) with \(Q_0 = 3, Q_1 = 2, Q_2 = 6\) in the last proposition, we have the following corollary which presents sum formulas of third-order Pell-Lucas numbers.

**Corollary 4.34.** For \(n \geq 0\), third-order Pell-Lucas numbers have the following properties:

(a): \(\sum_{k=0}^{n}(-1)^kQ_{k+1}^2 = \frac{1}{15}((-1)^n(Q_{n+3}^2+9Q_{n+2}^2+12Q_{n+1}+2Q_{n+2}Q_{n+1}+4Q_{n+2}Q_{n+1}-6Q_{n+3}Q_{n+1})+66).\)

(b): \(\sum_{k=0}^{n}(-1)^kQ_{k+1}Q_k = \frac{1}{5}((-1)^n(Q_{n+3}^2+Q_{n+2}^2-Q_{n+1}^2+3Q_{n+3}Q_{n+2}-Q_{n+3}Q_{n+1}+4Q_{n+2}Q_{n+1})+1).\)

(c): \(\sum_{k=0}^{n}(-1)^kQ_{k+2}Q_k = \frac{1}{15}((-1)^n(4Q_{n+3}^2-6Q_{n+2}^2-4Q_{n+2}^2+7Q_{n+3}Q_{n+2}+6Q_{n+3}Q_{n+1}+14Q_{n+2}Q_{n+1})+24).\)

From the last proposition, we have the following corollary which gives sum formulas of third-order modified Pell numbers (take \(W_n = E_n\) with \(E_0 = 0, E_1 = 1, E_2 = 1\)).

**Corollary 4.35.** For \(n \geq 0\), third-order modified Pell numbers have the following properties:

(a): \(\sum_{k=0}^{n}(-1)^kE_{k+1}^2 = \frac{1}{15}((-1)^n(E_{n+3}^2+9E_{n+2}^2+12E_{n+1}+2E_{n+2}E_{n+1}+4E_{n+2}E_{n+1}-6E_{n+3}E_{n+1})-6).\)

(b): \(\sum_{k=0}^{n}(-1)^kE_{k+1}E_k = \frac{1}{5}((-1)^n(E_{n+3}^2+E_{n+2}^2-E_{n+1}^2-3E_{n+3}E_{n+2}-E_{n+3}E_{n+1}+4E_{n+2}E_{n+1})-1).\)

(c): \(\sum_{k=0}^{n}(-1)^kE_{k+2}E_k = \frac{1}{15}((-1)^n(4E_{n+3}^2-6E_{n+2}^2-4E_{n+2}^2+7E_{n+3}E_{n+2}+6E_{n+3}E_{n+1}+14E_{n+2}E_{n+1})-9).\)

Taking \(r = 0, s = 1, t = 1\) in Theorem 3.1, we obtain the following proposition.

**Proposition 4.36.** If \(r = 0, s = 1, t = 1\) then for \(n \geq 0\) we have the following formulas:
ON THE SUMS OF SQUARES OF GENERALIZED TRIBONACCI NUMBERS:

(a): \( \sum_{k=0}^{n} (-1)^k W_k^2 = \frac{1}{5} \left( (1)^n \left( 3W_{n+3}^2 - 3W_{n+2}^2 + 2W_{n+1}^2 + 2W_{n+3}W_{n+2} - 2W_{n+3}W_{n+1} - 4W_{n+2}W_{n+1} \right) + 3W_{n+3}^2 - 3W_{n+2}^2 + 2W_{n+1}W_{n+1} - 2W_{n+2}W_{n+1} - 4W_{n+2}W_{n+1} \right) \).

(b): \( \sum_{k=0}^{n} (-1)^k W_{k+1}W_k = \frac{1}{5} \left( (1)^n \left( W_{n+3}^2 - W_{n+2}^2 - W_{n+1}^2 - W_{n+3}W_{n+2} + W_{n+3}W_{n+1} + 2W_{n+2}W_{n+1} \right) + W_{n+3}^2 - W_{n+2}^2 - W_{n+1}W_{n+1} + W_{n+2}W_{n+1} + 2W_{n+2}W_{n+1} \right) \).

(c): \( \sum_{k=0}^{n} (-1)^k W_{k+1}W_k = \frac{1}{5} \left( (1)^n \left( 2W_{n+3}^2 - 2W_{n+2}^2 - 2W_{n+1}^2 + 3W_{n+3}W_{n+2} + 2W_{n+3}W_{n+1} - 6W_{n+2}W_{n+1} + 2W_{n+2}^2 - 2W_{n+1}^2 - 2W_{n+1}W_{n+1} + 3W_{n+2}W_{n+1} - 6W_{n+2}W_{n+1} + 2W_{n+2}W_{n+1} \right) \).

From the last proposition, we have the following corollary which gives sum formulas of Padovan numbers (take \( W_n = P_n \) with \( P_0 = 1, P_1 = 1, P_2 = 1 \)).

**Corollary 4.37.** For \( n \geq 0 \), Padovan numbers have the following properties:

(a): \( \sum_{k=0}^{n} (-1)^k P_k^2 = \frac{1}{5} \left( (1)^n \left( 3P_{n+3}^2 - 3P_{n+2}^2 + 2P_{n+1}^2 + 2P_{n+3}P_{n+2} - 2P_{n+3}P_{n+1} - 4P_{n+2}P_{n+1} \right) - 2 \right) \).

(b): \( \sum_{k=0}^{n} (-1)^k P_{k+1}P_k = \frac{1}{5} \left( (1)^n \left( P_{n+3}^2 - P_{n+2}^2 - P_{n+1}^2 - P_{n+3}P_{n+2} - P_{n+3}P_{n+1} + 2P_{n+2}P_{n+1} \right) + 1 \right) \).

(c): \( \sum_{k=0}^{n} (-1)^k P_{k+2}P_k = \frac{1}{5} \left( (1)^n \left( 2P_{n+3}^2 - 2P_{n+2}^2 - 2P_{n+1}^2 + 3P_{n+3}P_{n+2} + 2P_{n+3}P_{n+1} - 6P_{n+2}P_{n+1} \right) - 3 \right) \).

Taking \( W_n = E_n \) with \( E_0 = 3, E_1 = 0, E_2 = 2 \) in the last proposition, we have the following corollary which presents sum formulas of Perrin numbers.

**Corollary 4.38.** For \( n \geq 0 \), Perrin numbers have the following properties:

(a): \( \sum_{k=0}^{n} (-1)^k E_k^2 = \frac{1}{5} \left( (1)^n \left( 3E_{n+3}^2 - 3E_{n+2}^2 + 2E_{n+1}^2 + 2E_{n+3}E_{n+2} - 2E_{n+3}E_{n+1} - 4E_{n+2}E_{n+1} \right) + 18 \right) \).

(b): \( \sum_{k=0}^{n} (-1)^k E_{k+1}E_k = \frac{1}{5} \left( (1)^n \left( E_{n+3}^2 - E_{n+2}^2 - E_{n+1}^2 - E_{n+3}E_{n+2} + E_{n+3}E_{n+1} + 2E_{n+2}E_{n+1} \right) + 1 \right) \).

(c): \( \sum_{k=0}^{n} (-1)^k E_{k+2}E_k = \frac{1}{5} \left( (1)^n \left( 2E_{n+3}^2 - 2E_{n+2}^2 - 2E_{n+1}^2 + 3E_{n+3}E_{n+2} + 2E_{n+3}E_{n+1} - 6E_{n+2}E_{n+1} \right) + 2 \right) \).

From the last proposition, we have the following corollary which gives sum formulas of Padovan-Perrin numbers (take \( W_n = S_n \) with \( S_0 = 0, S_1 = 0, S_2 = 1 \)).

**Corollary 4.39.** For \( n \geq 0 \), Padovan-Perrin numbers have the following properties:

(a): \( \sum_{k=0}^{n} (-1)^k S_k^2 = \frac{1}{5} \left( (1)^n \left( 3S_{n+3}^2 - 3S_{n+2}^2 + 2S_{n+1}^2 + 2S_{n+3}S_{n+2} - 2S_{n+3}S_{n+1} - 4S_{n+2}S_{n+1} \right) + 3 \right) \).

(b): \( \sum_{k=0}^{n} (-1)^k S_{k+1}S_k = \frac{1}{5} \left( (1)^n \left( S_{n+3}^2 - S_{n+2}^2 - S_{n+1}^2 - S_{n+3}S_{n+2} + S_{n+3}S_{n+1} + 2S_{n+2}S_{n+1} \right) + 1 \right) \).

(c): \( \sum_{k=0}^{n} (-1)^k S_{k+2}S_k = \frac{1}{5} \left( (1)^n \left( 2S_{n+3}^2 - 2S_{n+2}^2 - 2S_{n+1}^2 + 3S_{n+3}S_{n+2} + 2S_{n+3}S_{n+1} - 6S_{n+2}S_{n+1} \right) + 2 \right) \).

Taking \( r = 0, s = 2, t = 1 \) in Theorem 3.1, we obtain the following theorem.

**Theorem 4.40.** If \( r = 0, s = 2, t = 1 \) then for \( n \geq 0 \) we have the following formulas:

(a): \( \sum_{k=0}^{n} (-1)^k W_k^2 = \frac{1}{5} \left( (1)^n \left( 4n + 17 \right) W_{n+3}^2 - (4n + 13) W_{n+2}^2 - (4n + 11) W_{n+1}^2 + 4 (n + 5) W_{n+3}W_{n+2} - 4 (n + 6) W_{n+3}W_{n+1} - 4 (3n + 14) W_{n+2}W_{n+1} + 13W_{n+2}^2 - 9W_{n+1}^2 \right) - 4W_{n+3}W_{n+2} + 16W_{n+3}W_{n+1} + 16W_{n+2}W_{n+1} - 44W_{n+1}W_{n+1} \).
\( \sum_{k=0}^{n} (-1)^k W_{k+1} W_k = \frac{1}{10}((-1)^n (2n + 5) W_{n+3}^2 - 2(n + 4) W_{n+2}^2 - 2(n + 6) W_{n+1}^2 + (2n + 9) W_{n+3} W_{n+2} - (2n + 11) W_{n+3} W_{n+1} - (6n + 25) W_{n+2} W_{n+1} + 8W_{n+2}^2 - 6W_{n+1}^2 - 10W_{n+1}^2 + 7W_{n+2}^2 - 9W_{n+2} W_0 - 19W_1 W_0). \)

\( \sum_{k=0}^{n} (-1)^k W_{k+2} W_k = \frac{1}{10}((-1)^n (2n + 11) W_{n+3}^2 - 2(n + 8) W_{n+2}^2 - 2(n + 14) W_{n+1}^2 + (6n + 29) W_{n+3} W_{n+2} - (6n + 25) W_{n+3} W_{n+1} - (18n + 81) W_{n+2} W_{n+1} + 16W_{n+2}^2 - 10W_{n+1}^2 - 22W_{n+1}^2 + 23W_{n+2} W_1 - 19W_{n+2} W_0 - 63W_1 W_0). \)

Proof. The proof can be given exactly as in Theorem 4.12, just take \( x = -1 \) after using L'Hospital rule.

From the last theorem, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take \( W_n = R_n \) with \( Q_0 = 1, R_1 = 1, R_2 = 1 \)).

**Corollary 4.41.** For \( n \geq 0 \), Pell-Padovan numbers have the following properties:

- **(a):** \( \sum_{k=0}^{n} (-1)^k R_k^2 = \frac{1}{10}((-1)^n (4n + 17) R_{n+3}^2 - (4n + 13) R_{n+2}^2 - (4n + 11) R_{n+1}^2 + 4(n + 5) R_{n+3} R_{n+2} - 4(n + 6) R_{n+3} R_{n+1} - 4(3n + 14) R_{n+2} R_{n+1} - 51). \)
- **(b):** \( \sum_{k=0}^{n} (-1)^k R_k R_{k+1} = \frac{1}{10}((-1)^n (2(n + 5) R_{n+3}^2 - 2(n + 4) R_{n+2}^2 - 2(n + 6) R_{n+1}^2 + (2n + 9) R_{n+3} R_{n+2} - (2n + 11) R_{n+3} R_{n+1} - (6n + 25) R_{n+2} R_{n+1} - 29). \)
- **(c):** \( \sum_{k=0}^{n} (-1)^k R_{k+2} R_k = \frac{1}{10}((-1)^n (2(3n + 11) R_{n+3}^2 - 2(3n + 8) R_{n+2}^2 - 2(3n + 14) R_{n+1}^2 + (6n + 29) R_{n+3} R_{n+2} - (6n + 25) R_{n+3} R_{n+1} - (18n + 81) R_{n+2} R_{n+1} - 75). \)

Taking \( W_n = C_n \) with \( C_0 = 3, C_1 = 0, C_2 = 2 \) in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

**Corollary 4.42.** For \( n \geq 0 \), Pell-Perrin numbers have the following properties:

- **(a):** \( \sum_{k=0}^{n} (-1)^k C_k^2 = \frac{1}{10}((-1)^n (4n + 17) C_{n+3}^2 - (4n + 13) C_{n+2}^2 - (4n + 11) C_{n+1}^2 + 4(n + 5) C_{n+3} C_{n+2} - 4(n + 6) C_{n+2} C_{n+1} - 4(3n + 14) C_{n+2} C_{n+1} - 131). \)
- **(b):** \( \sum_{k=0}^{n} (-1)^k C_{k+1} C_k = \frac{1}{10}((-1)^n (2(n + 5) C_{n+3}^2 - 2(n + 4) C_{n+2}^2 - 2(n + 6) C_{n+1}^2 + (2n + 9) C_{n+3} C_{n+2} - (2n + 11) C_{n+3} C_{n+1} - (6n + 25) C_{n+2} C_{n+1} - 112). \)
- **(c):** \( \sum_{k=0}^{n} (-1)^k C_{k+2} C_k = \frac{1}{10}((-1)^n (2(3n + 11) C_{n+3}^2 - 2(3n + 8) C_{n+2}^2 - 2(3n + 14) C_{n+1}^2 + (6n + 29) C_{n+3} C_{n+2} - (6n + 25) C_{n+3} C_{n+1} - (18n + 81) C_{n+2} C_{n+1} - 248). \)

Taking \( r = 0, s = 1, t = 2 \) in Theorem 3.1, we obtain the following proposition.

**Proposition 4.43.** If \( r = 0, s = 1, t = 2 \) then for \( n \geq 0 \) we have the following formulas:

- **(a):** \( \sum_{k=0}^{n} (-1)^k W_k^2 = \frac{1}{10}((-1)^n (3W_{n+3}^2 - 3W_{n+2}^2 + 4W_{n+1}^2 + 2W_{n+3} W_{n+2} - 4W_{n+3} W_{n+1} - 4W_{n+2} W_{n+1}) + 3W_2^2 - 3W_1^2 + 4W_0^2 + 2W_2 W_1 - 4W_2 W_0 - 4W_1 W_0). \)
- **(b):** \( \sum_{k=0}^{n} (-1)^k W_{k+1} W_k = \frac{1}{10}((-1)^n (W_{n+3}^2 - W_{n+2}^2 - 4W_{n+1}^2 - 2W_{n+3} W_{n+2} + 4W_{n+3} W_{n+1} + 4W_{n+2} W_{n+1}) - 2W_2 W_1 + 4W_2 W_0 + 4W_1 W_0 + W_2^2 - W_1^2 - 4W_0^2). \)
- **(c):** \( \sum_{k=0}^{n} (-1)^k W_{k+2} W_k = \frac{1}{10}((-1)^n (W_{n+3}^2 - W_{n+2}^2 - 4W_{n+1}^2 + 6W_{n+3} W_{n+2} + 4W_{n+3} W_{n+1} - 12W_{n+2} W_{n+1}) + W_2^2 - W_1^2 - 4W_0^2 + 6W_2 W_1 + 4W_2 W_0 - 12W_1 W_0). \)
From the last proposition, we have the following corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

**Corollary 4.44.** For $n \geq 0$, Jacobsthal-Padovan numbers have the following properties:

(a): $\sum_{k=0}^{n}(-1)^k Q_k^2 = \frac{1}{10}((-1)^n (3Q_{n+3}^2 - 3Q_{n+2}^2 + 4Q_{n+1}^2 + 2Q_{n+3}Q_{n+2} - 4Q_{n+3}Q_{n+1} - 4Q_{n+2}Q_{n+1}) - 2)$.

(b): $\sum_{k=0}^{n}(-1)^k Q_{k+1}Q_k = \frac{1}{10}((-1)^n (Q_{n+3}^2 - Q_{n+2}^2 - 4Q_{n+1}^2 - 2Q_{n+3}Q_{n+2} + 4Q_{n+3}Q_{n+1} + 4Q_{n+2}Q_{n+1}) + 2)$.

(c): $\sum_{k=0}^{n}(-1)^k Q_{k+2}Q_k = \frac{1}{10}((-1)^n (Q_{n+3}^2 - Q_{n+2}^2 - 4Q_{n+1}^2 + 6Q_{n+3}Q_{n+2} + 4Q_{n+3}Q_{n+1} - 12Q_{n+2}Q_{n+1}) - 6)$.

Taking $W_n = D_n$ with $D_0 = 3, D_1 = 0, D_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Perrin numbers.

**Corollary 4.45.** For $n \geq 0$, Jacobsthal-Perrin numbers have the following properties:

(a): $\sum_{k=0}^{n}(-1)^k L_k^2 = \frac{1}{10}((-1)^n (3L_{n+3}^2 - 3L_{n+2}^2 + 4L_{n+1}^2 + 2L_{n+3}L_{n+2} - 4L_{n+3}L_{n+1} - 4L_{n+2}L_{n+1}) + 24)$.

(b): $\sum_{k=0}^{n}(-1)^k L_{k+1}L_k = \frac{1}{10}((-1)^n (L_{n+3}^2 - L_{n+2}^2 - 4L_{n+1}^2 - 2L_{n+3}L_{n+2} + 4L_{n+3}L_{n+1} + 4L_{n+2}L_{n+1}) - 8)$.

(c): $\sum_{k=0}^{n}(-1)^k L_{k+2}L_k = \frac{1}{10}((-1)^n (L_{n+3}^2 - L_{n+2}^2 - 4L_{n+1}^2 + 6L_{n+3}L_{n+2} + 4L_{n+3}L_{n+1} - 12L_{n+2}L_{n+1}) - 8)$.

Taking $r = 1, s = 0, t = 1$ in Theorem 3.1, we obtain the following proposition.

**Proposition 4.46.** If $r = 1, s = 0, t = 1$ then for $n \geq 0$ we have the following formulas:

(a): $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{3}((-1)^n (W_{n+3}^2 - 2W_{n+2}^2 + 2W_{n+1}^2 - 2W_{n+3}W_{n+1} + 2W_{n+2}W_{n+1}) + W_2^2 + 2W_0^2 - 2W_2W_0 + 2W_1W_0)$.

(b): $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{3}((-1)^n (W_{n+3}^2 + W_{n+2}^2 - W_{n+1}^2 - 3W_{n+3}W_{n+2} + W_{n+3}W_{n+1} + 2W_{n+2}W_{n+1}) + W_2^2 + W_0^2 - 3W_2W_0 + W_2W_0 + 2W_1W_0)$.

(c): $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k = \frac{1}{3}((-1)^n (-3W_{n+2}W_{n+1} + 3W_{n+3}W_{n+1}) + 3W_2W_0 - 3W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

**Corollary 4.47.** For $n \geq 0$, Narayana numbers have the following properties:

(a): $\sum_{k=0}^{n}(-1)^k N_k^2 = \frac{1}{3}((-1)^n (N_{n+3}^2 - 2N_{n+2}^2 + 2N_{n+1}^2 - 2N_{n+3}N_{n+1} + 2N_{n+2}N_{n+1}) - 1)$.

(b): $\sum_{k=0}^{n}(-1)^k N_{k+1}N_k = \frac{1}{3}((-1)^n (N_{n+3}^2 + N_{n+2}^2 - N_{n+1}^2 - 3N_{n+3}N_{n+2} + N_{n+3}N_{n+1} + 2N_{n+2}N_{n+1}) - 1)$.

(c): $\sum_{k=0}^{n}(-1)^k N_{k+2}N_k = \frac{1}{3}((-1)^n (-3N_{n+2}N_{n+1} + 3N_{n+3}N_{n+1})$.
Taking $W_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

**Corollary 4.48.** For $n \geq 0$, Narayana-Lucas numbers have the following properties:

(a): $\sum_{k=0}^{n} (-1)^k U_k^2 = \frac{1}{4}((-1)^n (U_{n+3}^2 - 2U_{n+2}^2 + 2U_{n+1}^2 - 2U_{n+3}U_{n+1} + 2U_{n+2}U_{n+1}) + 17)$.

(b): $\sum_{k=0}^{n} (-1)^k U_k + U_k = \frac{1}{3}((-1)^n (U_{n+3}^2 + U_{n+2}^2 - U_{n+1}^2 - 3U_{n+3}U_{n+2} + U_{n+3}U_{n+1} + 2U_{n+2}U_{n+1}) - 1)$.

(c): $\sum_{k=0}^{n} (-1)^k U_{k+2} U_k = \frac{1}{3}((-1)^n (-3U_{n+2}U_{n+1} + 3U_{n+3}U_{n+1}))$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $W_n = H_n$ with $H_0 = 3, H_1 = 0, H_2 = 2$).

**Corollary 4.49.** For $n \geq 0$, Narayana-Perrin numbers have the following properties:

(a): $\sum_{k=0}^{n} (-1)^k H_k^2 = \frac{1}{3}((-1)^n (H_{n+3}^2 - 2H_{n+2}^2 + 2H_{n+1}^2 - 2H_{n+3}H_{n+1} + 2H_{n+2}H_{n+1}) + 10)$.

(b): $\sum_{k=0}^{n} (-1)^k H_{k+1} H_k = \frac{1}{3}((-1)^n (H_{n+3}^2 + H_{n+2}^2 - H_{n+1}^2 - 3H_{n+3}H_{n+2} + H_{n+3}H_{n+1} + 2H_{n+2}H_{n+1}) + 1)$.

(c): $\sum_{k=0}^{n} (-1)^k H_{k+2} H_k = \frac{1}{3}((-1)^n (-3H_{n+2}H_{n+1} + 3H_{n+3}H_{n+1}) + 18)$.

Taking $r = 1, s = 1, t = 2$ in Theorem 3.1, we obtain the following proposition.

**Proposition 4.50.** If $r = 1, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:

(a): $\sum_{k=0}^{n} (-1)^k W_k^2 = \frac{1}{15}((-1)^n (2W_{n+3}^2 - 3W_{n+2}^2 + 7W_{n+1}^2 - W_{n+3}W_{n+2} - 6W_{n+3}W_{n+1} + 4W_{n+2}W_{n+1} + 2W_{n+2}^2 - 3W_{n+1}^2 + 7W_{n+1}^2 - W_{n+2}^2 - 6W_{n+2} - 4W_{n+1} + 4W_{n+2}))$.

(b): $\sum_{k=0}^{n} (-1)^k W_{k+1} W_k = \frac{1}{10}((-1)^n (W_{n+3}^2 + W_{n+2}^2 - 4W_{n+1}^1 - 3W_{n+3}W_{n+2} + 2W_{n+3}W_{n+1} + 2W_{n+2}W_{n+1} + W_{n+2}^2 + W_{n+1}^2 - 4W_{n+1}^2 - 3W_{n+1}W_{n+2} + 2W_{n+2}W_{n+1} + 2W_{n+1}))$.

(c): $\sum_{k=0}^{n} (-1)^k W_{k+2} W_k = \frac{1}{30}((-1)^n (W_{n+3}^2 - 9W_{n+2}^2 - 4W_{n+1}^2 - 7W_{n+3}W_{n+2} + 12W_{n+3}W_{n+1} - 28W_{n+2}W_{n+1} + W_{n+2}^2 - 9W_{n+1}^2 - 4W_{n+1}^2 + 7W_{n+1}W_{n+2} + 12W_{n+2}W_{n+1} - 28W_{n+1}))$.

From the above proposition, we have the following corollary which gives sum formulas of third order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$).

**Corollary 4.51.** For $n \geq 0$, third order Jacobsthal numbers have the following properties:

(a): $\sum_{k=0}^{n} (-1)^k J_k^2 = \frac{1}{15}((-1)^n (2J_{n+3}^2 - 3J_{n+2}^2 + 7J_{n+1}^2 - J_{n+3}J_{n+2} - 6J_{n+3}J_{n+1} + 4J_{n+2}J_{n+1} + 2J_{n+2}^2 - 3J_{n+1}^2 + 7J_{n+1}^2 - J_{n+2}^2 - 6J_{n+2} - 4J_{n+1} + 4J_{n+2}))$.

(b): $\sum_{k=0}^{n} (-1)^k J_{k+1} J_k = \frac{1}{10}((-1)^n (J_{n+3}^2 + J_{n+2}^2 - 4J_{n+1}^2 - 3J_{n+3}J_{n+2} + 2J_{n+3}J_{n+1} + 2J_{n+2}J_{n+1} - 1))$.

(c): $\sum_{k=0}^{n} (-1)^k J_{k+2} J_k = \frac{1}{30}((-1)^n (J_{n+3}^2 - 9J_{n+2}^2 - 4J_{n+1}^2 + 7J_{n+3}J_{n+2} + 12J_{n+3}J_{n+1} - 28J_{n+2}J_{n+1} - 1))$.

From the above proposition, we have the following corollary which gives sum formulas of third-order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$).

**Corollary 4.52.** For $n \geq 0$, third-order Jacobsthal-Lucas numbers have the following properties:
Taking \( r = 2, s = 3, t = 5 \) in Theorem 3.1, we obtain the following Proposition.

**Proposition 4.53.** If \( r = 2, s = 3, t = 5 \) then for \( n \geq 0 \) we have the following formulas:

(a): \[ \sum_{k=0}^{n} (-1)^k j^2_k = \frac{1}{180}((-1)^n (2j_{n+3}^2 - 3j_{n+2}^2 + 7j_{n+1}^2 - j_{n+3}j_{n+2} - 6j_{n+3}j_{n+1} + 4j_{n+2}j_{n+1}) + 18). \]

(b): \[ \sum_{k=0}^{n} (-1)^k j_{k+1}j_k = \frac{1}{180}((-1)^n (j_{n+3}^2 + j_{n+2}^2 - 4j_{n+1}j_{n+2} - 3j_{n+3}j_{n+2} + 2j_{n+3}j_{n+1} + 2j_{n+2}j_{n+1}) + 19). \]

(c): \[ \sum_{k=0}^{n} (-1)^k j_{k+2}j_k = \frac{1}{180}((-1)^n (j_{n+3}^2 - 9j_{n+2}^2 - 4j_{n+1}^2 + 7j_{n+3}j_{n+2} + 12j_{n+3}j_{n+1} - 28j_{n+2}j_{n+1}) + 99). \]

From the last proposition, we have the following corollary which gives sum formulas of 3-primes numbers (\( W_n = G_n \) with \( G_0 = 0, G_1 = 1, G_2 = 2 \)).

**Corollary 4.54.** For \( n \geq 0 \), 3-primes numbers have the following properties:

(a): \[ \sum_{k=0}^{n} (-1)^k G^2_k = \frac{1}{825}((-1)^n (19G_{n+3}^2 - 11G_{n+2}^2 + 350G_{n+1}^2 - 42G_{n+3}G_{n+2} - 170G_{n+3}G_{n+1} + 280 G_{n+2}G_{n+1}) + 19). \]

(b): \[ \sum_{k=0}^{n} (-1)^k G_{k+1}G_k = \frac{1}{825}((-1)^n (17G_{n+3}^2 + 77G_{n+2}^2 - 425G_{n+1}^2 - 81G_{n+3}G_{n+2} + 65G_{n+3}G_{n+1} + 10G_{n+2}G_{n+1}) + 17). \]

(c): \[ \sum_{k=0}^{n} (-1)^k G_{k+2}G_k = \frac{1}{825}((-1)^n (6G_{n+3}^2 - 264G_{n+2}^2 - 150G_{n+1}^2 + 117G_{n+3}G_{n+2} + 120G_{n+3}G_{n+1} - 780G_{n+2}G_{n+1}) - 6). \]

Taking \( W_n = H_n \) with \( H_0 = 3, H_1 = 2, H_2 = 10 \) in the last proposition, we have the following corollary which presents sum formulas of Lucas 3-primes numbers.

**Corollary 4.55.** For \( n \geq 0 \), Lucas 3-primes numbers have the following properties:

(a): \[ \sum_{k=0}^{n} (-1)^k H^2_k = \frac{1}{825}((-1)^n (19H_{n+3}^2 - 11H_{n+2}^2 + 350H_{n+1}^2 - 42H_{n+3}H_{n+2} - 170H_{n+3}H_{n+1} + 280 H_{n+2}H_{n+1}) + 746). \]

(b): \[ \sum_{k=0}^{n} (-1)^k H_{k+1}H_k = \frac{1}{825}((-1)^n (17H_{n+3}^2 + 77H_{n+2}^2 - 425H_{n+1}^2 - 81H_{n+3}H_{n+2} + 65H_{n+3}H_{n+1} - 10H_{n+2}H_{n+1}) - 1547). \]

(c): \[ \sum_{k=0}^{n} (-1)^k H_{k+2}H_k = \frac{1}{825}((-1)^n (6H_{n+3}^2 - 264H_{n+2}^2 - 150H_{n+1}^2 + 117H_{n+3}H_{n+2} + 120H_{n+3}H_{n+1} - 780H_{n+2}H_{n+1}) - 546). \]

From the last proposition, we have the following corollary which gives sum formulas of modified 3-primes numbers (\( W_n = E_n \) with \( E_0 = 0, E_1 = 1, E_2 = 1 \)).

**Corollary 4.56.** For \( n \geq 0 \), modified 3-primes numbers have the following properties:
(a): \( \sum_{k=0}^{n}(1-k)^2 E_k^2 = \frac{1}{825}((-1)^n (19E_{n+3}^2 - 11E_{n+2}^2 + 350E_{n+1}^2 - 42E_{n+3}E_{n+2} - 170E_{n+3}E_{n+1} + 280E_{n+2}E_{n+1}) - 34) \).

(b): \( \sum_{k=0}^{n}(1-k)^2 E_{k+1}^2 = \frac{1}{825}((-1)^n (17E_{n+3}^2 + 77E_{n+2}^2 - 425E_{n+1}^2 - 81E_{n+3}E_{n+2} + 65E_{n+3}E_{n+1} - 10E_{n+2}E_{n+1}) + 134) \).

(c): \( \sum_{k=0}^{n}(1-k)^2 E_{k+2}^2 = \frac{1}{825}((-1)^n (6E_{n+3}^2 - 264E_{n+2}^2 - 150E_{n+3}^2 + 117E_{n+3}E_{n+2} + 120E_{n+3}E_{n+1} - 780E_{n+2}E_{n+1}) - 141) \).

### 4.3. The Case \( x = 1+i \)

In this subsection we consider the special case \( x = 1+i \).

Taking \( x = 1+i, r = s = t = 1 \) in Theorem 3.1, we obtain the following Proposition.

**Proposition 4.57.** If \( x = 1+i, r = s = t = 1 \) then for \( n \geq 0 \) we have the following formulas:

(a): \( \sum_{k=0}^{n}(1+i)^k W_k^2 = \frac{1}{7+28i}((1+i)^n(6+14i)W_{n+3}^2 + (28+22i)E_{n+2}^2 + (27-i)E_{n+1}^2 + (-24-32i)W_{n+3}^2 - (-8-16i)W_{n+3}E_{n+1} + (-8+16i)W_{n+2}E_{n+1}) + (10+i)E_{n+1}^2 - (25-3i)W_{n+1}^2 - (13-14i)W_{n+2}^2 - (28+4i)W_{n+1}^2 + (12+4i)W_{n+0}^2 - (4+12i)W_{n+0}^2 \).

(b): \( \sum_{k=0}^{n}(1+i)^k W_{k+1}^2 = \frac{1}{7+28i}((1+i)^n(-6-2i)W_{n+3}^2 + (-16-8i)W_{n+2}^2 + (-4-8i)W_{n+1}^2 + (18+12i)W_{n+3}W_{n+2} + (-2+14i)W_{n+3}W_{n+1} + (11-25i)W_{n+2}W_{n+1} + (4-2i)W_{n+1}^2 + (12-4i)W_{n+2}^2 + (6+2i)W_{n+0}^2 + (15-3i)W_{n+1}^2 + (6+8i)W_{n+0}^2 + (7+18i)W_{n+0}^2 \).

(c): \( \sum_{k=0}^{n}(1+i)^k W_{k+2}^2 = \frac{1}{7+28i}((1+i)^n(-4+4i)W_{n+3}^2 + (4-10i)W_{n+2}^2 + (8W_{n+1}^2 + 10iW_{n+3}W_{n+2} + (9-11i)W_{n+3}W_{n+1} + (4-2i)W_{n+1}^2 + 4iW_{n+2}^2 + (4-4i)W_{n+1}^2 + (3+7i)W_{n+2}^2 + (5+5i)W_{n+1}^2 + (1+10i)W_{n+2}^0 + (-1+3i)W_{n+1}^0) \).

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take \( W_n = T_n \) with \( T_0 = 0, T_1 = 1, T_2 = 1 \)).

**Corollary 4.58.** For \( n \geq 0 \), Tribonacci numbers have the following properties:

(a): \( \sum_{k=0}^{n}(1+i)^k T_k^2 = \frac{1}{7+28i}((1+i)^n((6+14i)T_{n+3}^2 + (28+22i)T_{n+2}^2 + (27-i)T_{n+1}^2 + (-24-32i)T_{n+3}T_{n+2} + (-8-16i)T_{n+3}T_{n+1} + (-8+16i)T_{n+2}T_{n+1} - 7 + 3i) \).

(b): \( \sum_{k=0}^{n}(1+i)^k T_{k+1}^2 = \frac{1}{7+28i}((1+i)^n((-6-2i)T_{n+3}^2 + (-16-8i)T_{n+2}^2 + (-4-8i)T_{n+1}^2 + (18+12i)T_{n+3}T_{n+2} + (-2+14i)T_{n+3}T_{n+1} + (11-25i)T_{n+2}T_{n+1} + (4-2i)T_{n+1}^2 + (12-4i)T_{n+2}^2 + (6+2i)T_{n+0}^2 + (15-3i)T_{n+1}^2 + (6+8i)T_{n+0}^2 + (7+18i)T_{n+0}^2) \).

(c): \( \sum_{k=0}^{n}(1+i)^k T_{k+2}^2 = \frac{1}{7+28i}((1+i)^n((-4+4i)T_{n+3}^2 + (4-10i)T_{n+2}^2 - 8T_{n+1}^2 + 10iT_{n+3}T_{n+2} + (9-11i)T_{n+3}T_{n+1} + (4-2i)T_{n+1}^2 - 4iT_{n+2}^2 + (4-4i)T_{n+1}^2 + (3+7i)T_{n+2}^2 - (5+5i)T_{n+1}^2 + (1+10i)T_{n+2}^0 + (-1+3i)T_{n+1}^0) \).

Taking \( W_n = K_n \) with \( K_0 = 3, K_1 = 1, K_2 = 3 \) in the above proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

**Corollary 4.59.** For \( n \geq 0 \), Tribonacci-Lucas numbers have the following properties:

(a): \( \sum_{k=0}^{n}(1+i)^k K_k^2 = \frac{1}{7+28i}((1+i)^n((6+14i)K_{n+3}^2 + (28+22i)K_{n+2}^2 + (27-i)K_{n+1}^2 + (-24-32i)K_{n+3}K_{n+2} + (-8-16i)K_{n+3}K_{n+1} + (-8+16i)K_{n+2}K_{n+1} - 52 + 105i) \).
ON THE SUMS OF SQUARES OF GENERALIZED TRIBONACCI NUMBERS:

(b): \[ \sum_{k=0}^{n}(1+i)^k K_{k+1} K_k = \frac{1}{7+2\sqrt{i}} (1+i)^n ((-6+2i)K^2_{n+3} + (-16+8i)K^2_{n+2} + (-4+8i)K^2_{n+1} + (18+12i)K_{n+3}K_{n+2} + (-2+4i)K_{n+3}K_{n+1} + (11-25i)K_{n+2}K_{n+1} + 24-13i) + (24+13i). \]

(c): \[ \sum_{k=0}^{n}(1+i)^2 K_{k+2} K_k = \frac{1}{7+2\sqrt{i}} (1+i)^n ((-4+4i)K^2_{n+3} + (-10+4i)K^2_{n+2} - 8K^2_{n+1} + 10iK_{n+3}K_{n+2} + (9-11i)K_{n+3}K_{n+1} + (4-2i)K_{n+1}K_{n+2} + 30 + 19i). \]

Corresponding sums of the other third order generalized Tribonacci numbers can be calculated similarly.

5. Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Tribonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Tribonacci, Tribonacci-Lucas, Padovan, Perrin numbers and the other third order recurrence relations. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the squares of the numbers of the sequences. Our future work will be investigation of the closed forms of the sum formulas for the squares of generalized Tetranacci numbers.

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